Algebraic Logic, Quantum Algebra and Algebraic Mathematics

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Algebraic Logic, Quantum Logic, Quantum Algebra, Algebra, Algebraic Geometry, Algebraic Topology, Category Theory and Higher Dimensional Algebra v.2min

Boolean logic

Boolean logic is a complete system for logical operations, used in many systems. It was named after George Boole, who first defined an algebraic system of logic in the mid 19th century. Boolean logic has many applications in electronics, computer hardware and software, and is the basis of all modern digital electronics. In 1938, Claude Shannon showed how electric circuits with relays could be modeled with Boolean logic. This fact soon proved enormously consequential with the emergence of the electronic computer.

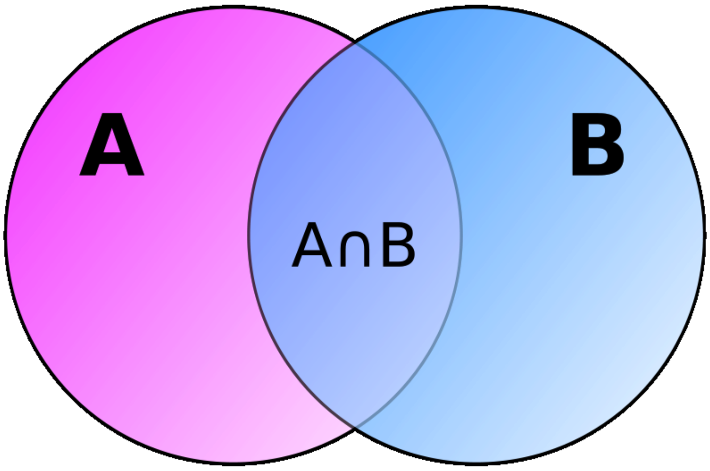
Using the algebra of sets, this article contains a basic introduction to sets, Boolean operations, Venn diagrams, truth tables, and Boolean applications. The Boolean algebra (structure) article discusses a type of algebraic structure that satisfies the axioms of Boolean logic. The binary arithmetic article discusses the use of binary numbers in computer systems.

Set logic vs. Boolean logic

Sets can contain any elements. We will first start out by discussing general set logic, then restrict ourselves to Boolean logic, where elements (or "bits") each contain only two possible values, called various names, such as "true" and "false", "yes" and "no", "on" and "off", or "1" and "0".

Terms

Let X be a set:



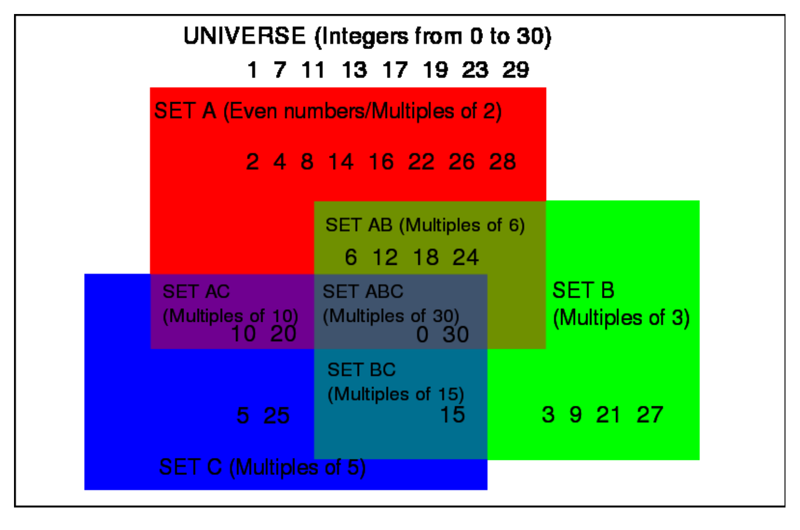
Venn diagram showing the intersection of sets "A AND B" (in violet/dark shading), the union of sets "A OR B" (all the colored regions), and the exclusive OR case "set A XOR B" (all the colored regions except the violet). The "universe" is represented by all the area within the rectangular frame.

- An **element** is one member of a set and is denoted by $\in$. If the element is not a member of a set it is denoted by $\notin$.
- The **universe** is the set X, sometimes denoted by 1. Note that this use of the word universe means *"all elements being considered"*, which are not necessarily the same as *"all elements there are"*.
- The **empty set** or **null set** is the set of no elements, denoted by $\varnothing$ and sometimes 0.
- A **unary operator** applies to a single set. There is only one unary operator, called logical **NOT**. It works by taking the complement with respect to the universe, i.e. the set of all elements under consideration.
- A **binary operator** applies to two sets. The basic binary operators are logical **OR** and logical **AND**. They perform the union and intersection of sets. There are also other derived binary operators, such as **XOR** (exclusive OR, i.e., "one or the other, but not both").
- A **subset** is denoted by $A \subseteq B$ and means every element in set A is also in set B.
- A **superset** is denoted by $A \supseteq B$ and means every element in set B is also in set A.
- The **identity** or **equivalence** of two sets is denoted by $A \equiv B$ and means that every element in set A is also in set B *and* every element in set B is also in set A.
- A **proper subset** is denoted by $A \subset B$ and means every element in set A is also in set B and the two sets are not identical.
- A **proper superset** is denoted by $A \supset B$ and means every element in set B is also in set A and the two sets are not identical.

Example

Imagine that set A contains all even numbers (multiples of two) in "the universe" (defined in the example below as all integers between 0 and 30 inclusive) and set B contains all multiples of three in "the universe". Then the **intersection** of the two sets (all elements in sets A AND B) would be all multiples of six in "the universe". The complement of set A (all elements NOT in set A) would be all odd numbers in "the universe".

Chaining operations together



While at most two sets are joined in any Boolean operation, the new set formed by that operation can then be joined with other sets utilizing additional Boolean operations. Using the previous example, we can define a new set C as the set of all multiples of five in "the universe". Thus "sets A AND B AND C" would be all multiples of 30 in "the universe". If more convenient, we may consider set AB to be the intersection of sets A and B, or the set of all multiples of six in "the universe". Then we can say "sets AB AND C" are the set of all multiples of 30 in "the universe". We could then take it a step further, and call this result set ABC.

Use of parentheses

While any number of logical ANDs (or any number of logical ORs) may be chained together without ambiguity, the combination of ANDs and ORs and NOTs can lead to ambiguous cases. In such cases, parentheses may be used to clarify the order of operations. As always, the operations within the innermost pair is performed first, followed by the next pair out, etc., until all operations within parentheses have been completed. Then any operations outside the parentheses are performed.

Application to binary values

In this example we have used natural numbers, while in Boolean logic binary numbers are used. The universe, for example, could contain just two elements, "1" and "0" (or "true" and "false", "yes" and "no", "on" or "off", etc.). We could also combine binary values together to get binary words, such as, in the case of two digits, "00", "01", "10", and "11". Applying set logic to those values, we could have a set of all values where the first digit is "0" ("00" and "01") and the set of all values where the first and second digits are different ("01" and "10"). The intersection of the two sets would then be the single element, "01". This could be shown by the following Boolean expression, where "1st" is the first digit and "2nd" is the second digit:

(NOT 1st) AND (1st XOR 2nd)

Properties

We define symbols for the two primary binary operations as $\wedge/\cap$ (logical AND/set intersection) and $\vee/\cup$ (logical OR/set union), and for the single unary operation $\neg$ / ~ (logical NOT/set complement). We will also use the values 0 (logical FALSE/the empty set) and 1 (logical TRUE/the universe). The following properties apply to both Boolean logic and set logic (although only the notation for Boolean logic is displayed here):

$a \vee (b \vee c) = (a \vee b) \vee c$	$a \wedge (b \wedge c) = (a \wedge b) \wedge c$	associativity
$a \vee b = b \vee a$	$a \wedge b = b \wedge a$	commutativity
$a \vee (a \wedge b) = a$	$a \wedge (a \vee b) = a$	absorption
$a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$	$a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$	distributivity
$a \vee \neg a = 1$	$a \wedge \neg a = 0$	complements
$a \vee a = a$	$a \wedge a = a$	idempotency
$a \vee 0 = a$	$a \wedge 1 = a$	boundedness
$a \vee 1 = 1$	$a \wedge 0 = 0$	
$\neg 0 = 1$	$\neg 1 = 0$	0 and 1 are complements
$\neg(a \vee b) = \neg a \wedge \neg b$	$\neg(a \wedge b) = \neg a \vee \neg b$	de Morgan's laws
$\neg\neg a = a$		involution

The first three properties define a lattice; the first five define a Boolean algebra. The remaining five are a consequence of the first five.

Other notations

Mathematicians and engineers often use plus (+) for OR and a product sign ($\cdot$) for AND. OR and AND are somewhat analogous to addition and multiplication in other algebraic structures, and this notation makes it very easy to get sum of products form for normal algebra. NOT may be represented by a line drawn above the expression being negated ($\overline{x}$). It also commonly leads to giving $\cdot$ a higher precedence than +, removing the need for parenthesis in some cases.

Programmers will often use a pipe symbol (|) for OR, an ampersand (&) for AND, and a tilde (~) for NOT. In many programming languages, these symbols stand for bitwise operations. "||", "&&", and "!" are used for variants of these operations.

Another notation uses "meet" for AND and "join" for OR. However, this can lead to confusion, as the term "join" is also commonly used for any Boolean operation which combines sets together, which includes both AND and OR.

Basic mathematics use of Boolean terms

- In the case of simultaneous equations, they are connected with an implied logical AND:

  x + y = 2

  AND

  x - y = 2

- The same applies to simultaneous inequalities:

  x + y < 2

  AND

  x - y < 2

- The greater than or equals sign ($\geq$) and less than or equals sign ($\leq$) may be assumed to contain a logical OR:

  X < 2

  OR

  X = 2
- The plus/minus sign ($\pm$), as in the case of the solution to a square root problem, may be taken as logical OR:

  WIDTH = 3

  OR

  WIDTH = -3

English language use of Boolean terms

Care should be taken when converting an English sentence into a formal boolean statement. Many English sentences have imprecise meanings.

- In certain cases, **AND** and **OR** can be used interchangeably in English: *I always carry an umbrella for when it rains* ***and*** *snows* has the same meaning as *I always carry an umbrella for when it rains* ***or*** *snows*. An alternate phrasing would be *I always carry an umbrella for when precipitation is forecast.*
- Sometimes the English words "and" and "or" have a meaning that is apparently opposite of its meaning in boolean logic: "Give me all the red **and** blue berries" usually means, "Give me all the berries that are **either** red **or** blue". An alternative phrasing for this request would be, "Give me all berries that are red and all berries that are blue."
- Depending on the context, the word "or" may correspond with either logical **OR** (which corresponds to the English equivalent "and/or") or logical **XOR** (which corresponds to the English equivalent "either/or"):
  - *The waitress asked, "Would you like cream* ***or*** *sugar with your coffee?"* This is an example of a "Logical **OR**", whereby the choices are cream, sugar, or cream and sugar (in addition to none of the above).
  - *The waitress asked, "Would you like soup* ***or*** *salad with your meal?"* This is an example of a "Logical **XOR**", whereby the choices are soup or salad (or neither), but soup **and** salad are not an option.)
  - This can be a significant challenge when providing precise specifications for a computer program or electronic circuit in English. The description of such functionality may be ambiguous. Take for example the statement, "The program should verify that the applicant has checked the male **or** female box." This is usually interpreted as an **XOR** and so a verification is performed to ensure that one, and only one, box is selected. In other cases the intended interpretation of English may be less obvious; the author of the specification should be consulted to determine the original intent.

Applications

Digital electronic circuit design

Boolean logic is also used for circuit design in electrical engineering; here 0 and 1 may represent the two different states of one bit in a digital circuit, typically high and low voltage. Circuits are described by expressions containing variables, and two such expressions are equal for all values of the variables if, and only if, the corresponding circuits have the same input-output behavior. Furthermore, every possible input-output behavior can be modeled by a suitable Boolean expression.

Basic logic gates such as AND, OR, and NOT gates may be used alone, or in conjunction with NAND, NOR, and XOR gates, to control digital electronics and circuitry. Whether these gates are wired in series or parallel controls the precedence of the operations.

Database applications

Relational databases use SQL, or other database-specific languages, to perform queries, which may contain Boolean logic. For this application, each record in a table may be considered to be an "element" of a "set". For example, in SQL, these SELECT statements are used to retrieve data from tables in the database:

```
SELECT * FROM employees WHERE last_name = 'Dean' AND first_name =
'James' ;
```

This example will produce a list of all employees, and only those employees, named James Dean.

```
SELECT * FROM employees WHERE last_name = 'Dean' OR  first_name =
'James' ;
```

This example will produce a list of all employees whose first name is James OR whose last name is Dean. Any and all employees named James Dean (from the first example) would also appear in this list.

```
SELECT * FROM employees WHERE NOT last_name = 'Dean' ;
```

This example will produce a list of all employees whose last name is not Dean. All employees named James from the second example would appear on this list, except for those employees named James Dean.

Parentheses may be used to explicitly specify the order in which Boolean operations occur, when multiple operations are present:

```
SELECT * FROM employees WHERE (NOT last_name = 'Smith') AND (first_name
 = 'John' OR first_name = 'Mary') ;
```

This example will produce a list of employees named John OR named Mary, but specifically excluding those named John Smith or Mary Smith.

Multiple sets of nested parentheses may also be used, where needed.

Any Boolean operation (or operations) which combines two (or more) tables together is referred to as a **join**, in relational database terminology.

In the field of Electronic Medical Records, some software applications use Boolean logic to query their patient databases, in what has been named Concept Processing technology.

Search engine queries

Search engine queries also employ Boolean logic. For this application, each web page on the Internet may be considered to be an "element" of a "set". The following examples use a syntax supported by Google.[1]

- Doublequotes are used to combine whitespace-separated words into a single search term.[2]
- Whitespace is used to specify logical AND, as it is the default operator for joining search terms:

```
"Search term 1" "Search term 2"
```

- The OR keyword is used for logical OR:

```
"Search term 1" OR "Search term 2"
```

- The minus sign is used for logical NOT (AND NOT):

```
"Search term 1" -"Search term 2"
```

Notes and references

[1] Not all search engines support the same query syntax. Additionally, some organizations provide "specialized" search engines that support alternate or extended syntax. (See e.g., Syntax cheatsheet (http://www.google.com/help/cheatsheet.html), Google codesearch supports regular expressions (http://www.google.com/intl/en/help/faq_codesearch.html#regexp)).

[2] Doublequote-delimited search terms are called "exact phrase" searches in the Google documentation.

External links

- The Calculus of Logic (http://www.maths.tcd.ie/pub/HistMath/People/Boole/CalcLogic/CalcLogic.html), by George Boole, Cambridge and Dublin Mathematical Journal Vol. III (1848), pp. 183–98.
- Logical Formula Evaluator (http://sourceforge.net/projects/logicaleval/) (for Windows), a software which calculates all possible values of a logical formula
- Maiki & Boaz BDD-PROJECT (http://www.bdd-project.com), a Web Application for BDD reduction and visualization.

Intuitionistic logic

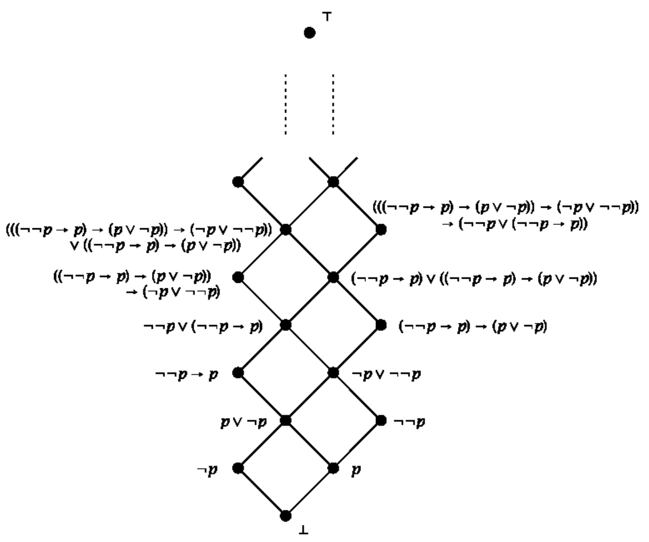
Intuitionistic logic, or **constructive logic**, is a symbolic logic system that differs from classical logic in its definition of what it means for a statement to be true. In classical logic, all well-formed statements are assumed to be either true or false, even if we do not have a proof of either. In constructive logic, a statement is only true if there is a proof that it is true, and only false if there is a proof that it is false. Operations in constructive logic preserve justification, rather than truth. Syntactically, intuitionist logic differs from classical logic in that the law of excluded middle and double negation elimination are not axioms of the system, and cannot be proved in it.

Constructive logic is practically useful because its restrictions produce proofs that have the existence property, making it also suitable for other forms of mathematical constructivism. Informally, this means that if you have a constructive proof that an object exists, you can turn that constructive proof into an algorithm for generating an example of it.

It was originally developed by Arend Heyting to provide a formal basis for Brouwer's programme of intuitionism.

Syntax

The syntax of formulas of intuitionistic logic is similar to propositional logic or first-order logic. However, intuitionistic connectives are not definable in terms of each other in the same way as in classical logic, hence their choice matters. In intuitionistic propositional logic it is customary to use →, ∧, ∨, ⊥ as the basic connectives, treating ¬*A* as an abbreviation for (*A* → ⊥). In intuitionistic first-order logic both quantifiers ∃, ∀ are needed.



The Rieger–Nishimura lattice. Its nodes are the propositional formulas in one variable up to intuitionistic logical equivalence, ordered by intuitionistic logical implication.

Many tautologies of classical logic can no longer be proven within intuitionistic logic. Examples include not only the law of excluded middle *p* ∨ ¬*p*, but also Peirce's law ((*p* → *q*) → *p*) → *p*, and even double negation elimination. In classical logic, both *p* → ¬¬*p* and also ¬¬*p* → *p* are theorems. In intuitionistic logic, only the former is a theorem: double negation can be introduced, but it cannot be eliminated. Rejecting *p* ∨ ¬*p* may seem strange to those more familiar with classical logic, but proving this statement in constructive logic would require producing a proof for the truth or falsity of *all possible statements*, which is impossible for a variety of reasons.

Because many classically valid tautologies are not theorems of intuitionistic logic, but all theorems of intuitionist logic are valid classically, intuitionist logic can be viewed as a weakening of classical logic, albeit one with many useful properties.

Sequent calculus

Gentzen discovered that a simple restriction of his system LK (his sequent calculus for classical logic) results in a system which is sound and complete with respect to intuitionistic logic. He called this system LJ. In LK any number of formulas is allowed to appear on the conclusion side of a sequent; in contrast LJ allows at most one formula in this position.

Other derivatives of LK are limited to intuitionstic derivations but still allow multiple conclusions in a sequent. LJ'[1] is one example.

Hilbert-style calculus

Intuitionistic logic can be defined using the following Hilbert-style calculus. Compare with the deduction system at Propositional calculus#Alternative calculus.

In propositional logic, the inference rule is modus ponens

- MP: from ϕ and $\phi \to \psi$ infer ψ

and the axioms are

- THEN-1: $\phi \to (\chi \to \phi)$
- THEN-2: $(\phi \to (\chi \to \psi)) \to ((\phi \to \chi) \to (\phi \to \psi))$
- AND-1: $\phi \land \chi \to \phi$
- AND-2: $\phi \land \chi \to \chi$

- AND-3: $\phi \to (\chi \to (\phi \wedge \chi))$
- OR-1: $\phi \to \phi \vee \chi$
- OR-2: $\chi \to \phi \vee \chi$
- OR-3: $(\phi \to \psi) \to ((\chi \to \psi) \to (\phi \vee \chi \to \psi))$
- FALSE: $\bot \to \phi$

To make this a system of first-order predicate logic, the generalization rules

- $\forall$ -GEN: from $\psi \to \phi$ infer $\psi \to (\forall x\ \phi)$, if x is not free in ψ
- $\exists$ -GEN: from $\phi \to \psi$ infer $(\exists x\ \phi) \to \psi$, if x is not free in ψ

are added, along with the axioms

- PRED-1: $(\forall x\ \phi(x)) \to \phi(t)$, if the term *t* is free for substitution for the variable *x* in ϕ (i.e., if no occurrence of any variable in *t* becomes bound in $\phi(t)$)
- PRED-2: $\phi(t) \to (\exists x\ \phi(x))$, with the same restriction as for PRED-1

Optional connectives

Negation

If one wishes to include a connective $\neg$ for negation rather than consider it an abbreviation for $\phi \to \bot$, it is enough to add:

- NOT-1': $(\phi \to \bot) \to \neg\phi$
- NOT-2': $\neg\phi \to (\phi \to \bot)$

There are a number of alternatives available if one wishes to omit the connective $\bot$ (false). For example, one may replace the three axioms FALSE, NOT-1', and NOT-2' with the two axioms

- NOT-1: $(\phi \to \chi) \to ((\phi \to \neg\chi) \to \neg\phi)$
- NOT-2: $\phi \to (\neg\phi \to \chi)$

as at Propositional calculus#Axioms. Alternatives to NOT-1 are $(\phi \to \neg\chi) \to (\chi \to \neg\phi)$ or $(\phi \to \neg\phi) \to \neg\phi$.

Equivalence

The connective $\leftrightarrow$ for equivalence may be treated as an abbreviation, with $\phi \leftrightarrow \chi$ standing for $(\phi \to \chi) \wedge (\chi \to \phi)$. Alternatively, one may add the axioms

- IFF-1: $(\phi \leftrightarrow \chi) \to (\phi \to \chi)$
- IFF-2: $(\phi \leftrightarrow \chi) \to (\chi \to \phi)$
- IFF-3: $(\phi \to \chi) \to ((\chi \to \phi) \to (\phi \leftrightarrow \chi))$

IFF-1 and IFF-2 can, if desired, be combined into a single axiom $(\phi \leftrightarrow \chi) \to ((\phi \to \chi) \wedge (\chi \to \phi))$ using conjunction.

Relation to classical logic

The system of classical logic is obtained by adding any one of the following axioms:

- $\phi \vee \neg\phi$ (Law of the excluded middle. May also be formulated as $(\phi \to \chi) \to ((\neg\phi \to \chi) \to \chi)$.)
- $\neg\neg\phi \to \phi$ (Double negation elimination)
- $((\phi \to \chi) \to \phi) \to \phi$ (Peirce's law)

In general, one may take as the extra axiom any classical tautology that is not valid in the two-element Kripke frame $\circ \longrightarrow \circ$ (in other words, that is not included in Smetanich's logic).

Another relationship is given by the Gödel–Gentzen negative translation, which provides an embedding of classical first-order logic into intuitionistic logic: a first-order formula is provable in classical logic if and only if its Gödel–Gentzen translation is provable intuitionistically. Therefore intuitionistic logic can instead be seen as a means of extending classical logic with constructive semantics.

Non-interdefinability of operators

In classical propositional logic, it is possible to take one of conjunction, disjunction, or implication as primitive, and define the other two in terms of it together with negation, such as in Łukasiewicz's three axioms of propositional logic. It is even possible to define all four in terms of a sole sufficient operator such as the Peirce arrow (NOR) or Sheffer stroke (NAND). Similarly, in classical first-order logic, one of the quantifiers can be defined in terms of the other and negation.

These are fundamentally consequences of the law of bivalence, which makes all such connectives merely Boolean functions. The law of bivalence does not hold in intuitionistic logic, only the law of non-contradiction. As a result none of the basic connectives can be dispensed with, and the above axioms are all necessary. Most of the classical identities are only theorems of intuitionistic logic in one direction, although some are theorems in both directions. They are as follows:

Conjunction versus disjunction:

- $(\phi \wedge \psi) \to \neg(\neg\phi \vee \neg\psi)$
- $(\phi \vee \psi) \to \neg(\neg\phi \wedge \neg\psi)$
- $(\neg\phi \vee \neg\psi) \to \neg(\phi \wedge \psi)$
- $(\neg\phi \wedge \neg\psi) \leftrightarrow \neg(\phi \vee \psi)$

Conjunction versus implication:

- $(\phi \wedge \psi) \to \neg(\phi \to \neg\psi)$
- $(\phi \to \psi) \to \neg(\phi \wedge \neg\psi)$
- $(\phi \wedge \neg\psi) \to \neg(\phi \to \psi)$
- $(\phi \to \neg\psi) \leftrightarrow \neg(\phi \wedge \psi)$

Disjunction versus implication:

- $(\phi \vee \psi) \to (\neg\phi \to \psi)$
- $(\neg\phi \vee \psi) \to (\phi \to \psi)$
- $\neg(\phi \to \psi) \to \neg(\neg\phi \vee \psi)$
- $\neg(\phi \vee \psi) \leftrightarrow \neg(\neg\phi \to \psi)$

Universal versus existential quantification:

- $(\forall x\ \phi(x)) \to \neg(\exists x\ \neg\phi(x))$
- $(\exists x\ \phi(x)) \to \neg(\forall x\ \neg\phi(x))$
- $(\exists x\ \neg\phi(x)) \to \neg(\forall x\ \phi(x))$
- $(\forall x\ \neg\phi(x)) \leftrightarrow \neg(\exists x\ \phi(x))$

So, for example, "a or b" is a stronger statement than "if not a, then b", whereas these are classically interchangeable. On the other hand, "neither a nor b" is equivalent to "not a, and also not b".

If we include equivalence in the list of connectives, some of the connectives become definable from others:

- $(\phi \leftrightarrow \psi) \leftrightarrow ((\phi \to \psi) \land (\psi \to \phi))$
- $(\phi \to \psi) \leftrightarrow ((\phi \lor \psi) \leftrightarrow \psi)$
- $(\phi \to \psi) \leftrightarrow ((\phi \land \psi) \leftrightarrow \phi)$
- $(\phi \land \psi) \leftrightarrow ((\phi \to \psi) \leftrightarrow \phi)$
- $(\phi \land \psi) \leftrightarrow (((\phi \lor \psi) \leftrightarrow \psi) \leftrightarrow \phi)$

In particular, $\{\lor, \leftrightarrow, \bot\}$ and $\{\lor, \leftrightarrow, \neg\}$ are complete bases of intuitionistic connectives.

As shown by Alexander Kuznetsov, either of the following defined connectives can serve the role of a sole sufficient operator for intuitionistic logic:[2]

- $((p \lor q) \land \neg r) \lor (\neg p \land (q \leftrightarrow r)),$
- $p \to (q \land \neg r \land (s \lor t)).$

Semantics

The semantics are rather more complicated than for the classical case. A model theory can be given by Heyting algebras or, equivalently, by Kripke semantics.

Heyting algebra semantics

In classical logic, we often discuss the truth values that a formula can take. The values are usually chosen as the members of a Boolean algebra. The meet and join operations in the Boolean algebra are identified with the $\land$ and $\lor$ logical connectives, so that the value of a formula of the form $A \land B$ is the meet of the value of A and the value of B in the Boolean algebra. Then we have the useful theorem that a formula is a valid sentence of classical logic if and only if its value is 1 for every valuation—that is, for any assignment of values to its variables.

A corresponding theorem is true for intuitionistic logic, but instead of assigning each formula a value from a Boolean algebra, one uses values from a Heyting algebra, of which Boolean algebras are a special case. A formula is valid in intuitionistic logic if and only if it receives the value of the top element for any valuation on any Heyting algebra.

It can be shown that to recognize valid formulas, it is sufficient to consider a single Heyting algebra whose elements are the open subsets of the real line **R**.[3] In this algebra, the $\land$ and $\lor$ operations correspond to set intersection and union, and the value assigned to a formula $A \to B$ is $\mathrm{int}(A^C \cup B)$, the interior of the union of the value of B and the complement of the value of A. The bottom element is the empty set $\varnothing$, and the top element is the entire line **R**. The negation $\neg A$ of a formula A is (as usual) defined to be $A \to \varnothing$. The value of $\neg A$ then reduces to $\mathrm{int}(A^C)$, the interior of the complement of the value of A, also known as the exterior of A. With these assignments, intuitionistically valid formulas are precisely those that are assigned the value of the entire line.[3]

For example, the formula $\neg(A \land \neg A)$ is valid, because no matter what set X is chosen as the value of the formula A, the value of $\neg(A \land \neg A)$ can be shown to be the entire line:

$\mathrm{Value}(\neg(A \land \neg A)) =$

$\mathrm{int}((\mathrm{Value}(A \land \neg A))^C) =$

$\mathrm{int}((\mathrm{Value}(A) \cap \mathrm{Value}(\neg A))^C) =$

$\mathrm{int}((X \cap \mathrm{int}((\mathrm{Value}(A))^C))^C) =$

$\mathrm{int}((X \cap \mathrm{int}(X^C))^C)$

A theorem of topology tells us that $\mathrm{int}(X^C)$ is a subset of X^C, so the intersection is empty, leaving:

$\mathrm{int}(\varnothing^C) = \mathrm{int}(\mathbf{R}) = \mathbf{R}$

So the valuation of this formula is true, and indeed the formula is valid.

But the law of the excluded middle, $A \vee \neg A$, can be shown to be *invalid* by letting the value of A be $\{y : y > 0\}$. Then the value of $\neg A$ is the interior of $\{y : y \leq 0\}$, which is $\{y : y < 0\}$, and the value of the formula is the union of $\{y : y > 0\}$ and $\{y : y < 0\}$, which is $\{y : y \neq 0\}$, *not* the entire line.

The interpretation of any intuitionistically valid formula in the infinite Heyting algebra described above results in the top element, representing true, as the valuation of the formula, regardless of what values from the algebra are assigned to the variables of the formula.[3] Conversely, for every invalid formula, there is an assignment of values to the variables that yields a valuation that differs from the top element.[4] [5] No finite Heyting algebra has both these properties.[3]

Kripke semantics

Building upon his work on semantics of modal logic, Saul Kripke created another semantics for intuitionistic logic, known as **Kripke semantics** or **relational semantics**.[6]

Relation to other logics

Intutionistic logic is related by duality to a paraconsistent logic known as *Brazilian*, *anti-intuitionistic* or *dual-intuitionistic logic*.[7]

The subsystem of intuitionistic logic with the FALSE axiom removed is known as minimal logic.

Notes

[1] Proof Theory by G. Takeuti, ISBN0444104925

[2] Alexander Chagrov, Michael Zakharyaschev, *Modal Logic*, vol. 35 of Oxford Logic Guides, Oxford University Press, 1997, pp. 58–59.

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External links

- Stanford Encyclopedia of Philosophy: " Intuitionistic Logic (http://plato.stanford.edu/entries/logic-intuitionistic/)" -- by Joan Moschovakis.

Heyting arithmetic

In mathematical logic, **Heyting arithmetic** (sometimes abbreviated HA) is an axiomatization of arithmetic in accordance with the philosophy of intuitionism. It is named after Arend Heyting, who first proposed it.

Heyting arithmetic adopts the axioms of Peano arithmetic (PA), but uses intuitionistic logic as its rules of inference. In particular, the law of the excluded middle does not hold in general, though the induction axiom can be used to prove many specific cases. For instance, one can prove that $\forall\ x, y \in \mathbf{N} : x = y \vee x \neq y$ is a theorem (any two natural numbers are either equal to each other, or not equal to each other). In fact, since "=" is the only predicate symbol in Heyting arithmetic, it then follows that, for any quantifier-free formula p, $\forall\ x, y, z, \ldots \in \mathbf{N} : p \vee \neg p$ is a theorem (where $x,y,z\ldots$ are the free variables in p).

Kurt Gödel studied the relationship between Heyting arithmetic and Peano arithmetic. He used the Gödel–Gentzen negative translation to prove in 1933 that if HA is consistent, then PA is also consistent.

Heyting arithmetic should not be confused with Heyting algebras, which are the intuitionistic analogue of Boolean algebras.

External links

- Stanford Encyclopedia of Philosophy: "Intuitionistic Number Theory [1]" by Joan Moschovakis.

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Algebraic Logic and Many-Valued Logic

Algebraic logic

In mathematical logic, **algebraic logic** is the study of logic presented in an algebraic style.

Algebras as models of logics

Algebraic logic treats algebraic structures, often bounded lattices, as models (interpretations) of certain logics, making logic a branch of order theory.

In algebraic logic:

- Variables are tacitly universally quantified over some universe of discourse. There are no existentially quantified variables or open formulas;
- Terms are built up from variables using primitive and defined operations. There are no connectives;
- Formulas, built from terms in the usual way, can be equated if they are logically equivalent. To express a tautology, equate a formula with a truth value;
- The rules of proof are the substitution of equals for equals, and uniform replacement. Modus ponens remains valid, but is seldom employed.

In the table below, the left column contains one or more logical or mathematical systems, and the algebraic structure which are its models are shown on the right in the same row. Some of these structures are either Boolean algebras or proper extensions thereof. Modal and other nonclassical logics are typically modeled by what are called "Boolean algebras with operators."

Algebraic formalisms going beyond first-order logic in at least some respects include:

- Combinatory logic, having the expressive power of set theory;
- Relation algebra, arguably the paradigmatic algebraic logic, can express Peano arithmetic and most axiomatic set theories, including the canonical ZFC.

logical system	**its models**
Classical sentential logic	Lindenbaum-Tarski algebra Two-element Boolean algebra
Intuitionistic propositional logic	Heyting algebra
Łukasiewicz logic	MV-algebra
Modal logic K	Modal algebra
Lewis's S4	Interior algebra
Lewis's S5; Monadic predicate logic	Monadic Boolean algebra
First-order logic	Cylindric algebra Polyadic algebra Predicate functor logic
Set theory	Combinatory logic Relation algebra

History

On the history of algebraic logic before World War II, see Brady (2000) and Grattan-Guinness (2000) and their ample references. On the postwar history, see Maddux (1991) and Quine (1976).

Algebraic logic has at least two meanings:

- The study of Boolean algebra, begun by George Boole, and of relation algebra, begun by Augustus DeMorgan, extended by Charles Sanders Peirce, and taking definitive form in the work of Ernst Schröder;
- Abstract algebraic logic, a branch of contemporary mathematical logic.

Perhaps surprisingly, algebraic logic is the oldest approach to formal logic, arguably beginning with a number of memoranda Leibniz wrote in the 1680s, some of which were published in the 19th century and translated into English by Clarence Lewis in 1918. But nearly all of Leibniz's known work on algebraic logic was published only in 1903, after Louis Couturat discovered it in Leibniz's Nachlass. Parkinson (1966) and Loemker (1969) translated selections from Couturat's volume into English.

Brady (2000) discusses the rich historical connections between algebraic logic and model theory. The founders of model theory, Ernst Schroder and Leopold Loewenheim, were logicians in the algebraic tradition. Alfred Tarski, the founder of set theoretic model theory as a major branch of contemporary mathematical logic, also:

- Co-discovered Lindenbaum-Tarski algebra;
- Invented cylindric algebra;
- Wrote the 1941 paper that revived relation algebra, and that can be seen as the starting point of abstract algebraic logic.

Modern mathematical logic began in 1847, with two pamphlets whose respective authors were Augustus DeMorgan and George Boole. They, and later C.S. Peirce, Hugh MacColl, Frege, Peano, Bertrand Russell, and A. N. Whitehead all shared Leibniz's dream of combining symbolic logic, mathematics, and philosophy. Relation algebra is arguably the culmination of Leibniz's approach to logic. With the exception of some writings by Leopold Loewenheim and Thoralf Skolem, algebraic logic went into eclipse soon after the 1910-13 publication of *Principia Mathematica*, not to revive until Tarski's 1940 reexposition of relation algebra.

Leibniz had no influence on the rise of algebraic logic because his logical writings were little studied before the Parkinson and Loemker translations. Our present understanding of Leibniz the logician stems mainly from the work of Wolfgang Lenzen, summarized in Lenzen (2004). [1] To see how present-day work in logic and metaphysics can draw inspiration from, and shed light on, Leibniz's thought, see Zalta (2000). [2]

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Łukasiewicz logic

In mathematics, **Łukasiewicz logic** (English pronunciation: /luːkəˈʃɛvɪtʃ/, Polish pronunciation: [wukaˈɕɛvʲitʂ]) is a non-classical, many valued logic. It was originally defined in the early 20th-century by Jan Łukasiewicz as a three-valued logic;[1] it was later generalized to *n*-valued (for all finite *n*) as well as infinitely-many-valued variants, both propositional and first-order.[2] It belongs to the classes of t-norm fuzzy logics[3] and substructural logics.[4]

Language

The propositional connectives of Łukasiewicz logic are *implication* $\rightarrow$, *negation* $\neg$, *equivalence* $\leftrightarrow$, *weak conjunction* $\wedge$, *strong conjunction* $\otimes$, *weak disjunction* $\vee$, *strong disjunction* $\oplus$, and propositional constants $\overline{0}$ and $\overline{1}$. The presence of weak and strong conjunction and disjunction is a common feature of substructural logics without the rule of contraction, among which Łukasiewicz logic belongs.

Axioms

The original system of axioms for propositional infinite-valued Łukasiewicz logic used implication and negation as the primitive connectives:

$$A \rightarrow (B \rightarrow A)$$
$$(A \rightarrow B) \rightarrow ((B \rightarrow C) \rightarrow (A \rightarrow C))$$
$$((A \rightarrow B) \rightarrow B) \rightarrow ((B \rightarrow A) \rightarrow A)$$
$$(\neg B \rightarrow \neg A) \rightarrow (A \rightarrow B).$$

Propositional infinite-valued Łukasiewicz logic can also be axiomatized by adding the following axioms to the axiomatic system of monoidal t-norm logic:

- *Divisibility:* $(A \wedge B) \rightarrow (A \otimes (A \rightarrow B))$
- *Double negation:* $\neg\neg A \rightarrow A.$

That is, infinite-valued Łukasiewicz logic arises by adding the axiom of double negation to basic t-norm logic BL, or by adding the axiom of divisibility to the logic IMTL.

Real-valued semantics

Infinite-valued Łukasiewicz logic is a real-valued logic in which sentences from sentential calculus may be assigned a truth value of not only zero or one but also any real number in between (eg. 0.25). Valuations have a recursive definition where:

- $w(\theta \circ \phi) = F_\circ(w(\theta), w(\phi))$ for a binary connective $\circ$,
- $w(\neg\theta) = F_\neg(w(\theta))$,
- $w(\overline{0}) = 0$ and $w(\overline{1}) = 1$,

and where the definitions of the operations hold as follows:

- **Implication:** $F_\rightarrow(x, y) = \min\{1, 1 - x + y\}$
- **Equivalence:** $F_\leftrightarrow(x, y) = 1 - |x - y|$
- **Negation:** $F_\neg(x) = 1 - x$
- **Weak Conjunction:** $F_\wedge(x, y) = \min\{x, y\}$
- **Weak Disjunction:** $F_\vee(x, y) = \max\{x, y\}$
- **Strong Conjunction:** $F_\otimes(x, y) = \max\{0, x + y - 1\}$
- **Strong Disjunction:** $F_\oplus(x, y) = \min\{1, x + y\}$.

The truth function $F_\otimes$ of strong conjunction is the Łukasiewicz t-norm and the truth function $F_\oplus$ of strong disjunction is its dual t-conorm. The truth function $F_\rightarrow$ is the residuum of the Łukasiewicz t-norm. All truth functions of the basic connectives are continuous.

By definition, a formula is a tautology of infinite-valued Łukasiewicz logic if it evaluates to 1 under any valuation of propositional variables by real numbers in the interval [0, 1].

General algebraic semantics

The standard real-valued semantics determined by the Łukasiewicz t-norm is not the only possible semantics of Łukasiewicz logic. General algebraic semantics of propositional infinite-valued Łukasiewicz logic is formed by the class of all MV-algebras. The standard real-valued semantics is a special MV-algebra, called the *standard MV-algebra*.

Like other t-norm fuzzy logics, propositional infinite-valued Łukasiewicz logic enjoys completeness with respect to the class of all algebras for which the logic is sound (that is, MV-algebras) as well as with respect to only linear ones. This is expressed by the general, linear, and standard completeness theorems:

The following conditions are equivalent:

- A is provable in propositional infinite-valued Łukasiewicz logic
- A is valid in all MV-algebras (*general completeness*)
- A is valid in all linearly ordered MV-algebras (*linear completeness*)
- A is valid in the standard MV-algebra (*standard completeness*).

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Ternary logic

A **ternary**, **three-valued** or **trivalent logic** (sometimes abbreviated **3VL**) is any of several multi-valued logic systems in which there are three truth values indicating *true*, *false* and some indeterminate third value. This is contrasted with the more commonly known bivalent logics (such as classical sentential or boolean logics) which provide only for *true* and *false*. Conceptual form and basic ideas were initially created by Łukasiewicz, Lewis and Sulski. These were then re-formulated by Grigore Moisil in an axiomatic algebraic form, and also extended to *n*-valued logics in 1945.

Definitions

Concerning fuzziness, ternary logic might be seen formally as a fuzzy type of logic as a value may be different from just false (0) or true (1); however, ternary logic is defined as a *crisp logic*.

Representation of values

As with bivalent logic, truth values in ternary logic may be represented numerically using various representations of the ternary numeral system. A few of the more common examples are:

- 1 for *true*, 2 for *false*, and 0 for *unknown*, *irrelevant*, or *both*.[1]
- 0 for *false*, 1 for *true*, and a third non-integer symbol such as # or ½ for the final value.[2]
- Balanced ternary uses −1 for *false*, +1 for *true* and 0 for the third value; these values may also be simplified to −, +, and 0, respectively.[3]

This article mainly illustrates a system of ternary propositional logic using the truth values {*false, unknown,* and *true*}, and extends conventional boolean connectives to a trivalent context. Ternary predicate logics exist as well; these may have readings of the quantifier different from classical (binary) predicate logic, and may include alternative quantifiers as well.

Basic truth table

Below is a truth table showing the logic operations for Kleene's logic.

A	*B*	*A* OR *B*	*A* AND *B*	NOT *A*
True	True	True	True	False
True	Unknown	True	Unknown	False
True	False	True	False	False
Unknown	True	True	Unknown	Unknown
Unknown	Unknown	Unknown	Unknown	Unknown
Unknown	False	Unknown	False	Unknown
False	True	True	False	True
False	Unknown	Unknown	False	True
False	False	False	False	True

In this truth table, the UNKNOWN state can be metaphorically thought of as a sealed box containing either an unambiguously TRUE or unambiguously FALSE value. The knowledge of whether any particular UNKNOWN state secretly represents TRUE or FALSE at any moment in time is not available. However, certain logical operations can yield an unambiguous result, even if they involve at least one UNKNOWN operand. For example, since TRUE OR TRUE equals TRUE, and TRUE OR FALSE also equals TRUE, one can infer that TRUE OR UNKNOWN equals

TRUE, as well. In this example, since either bivalent state could be underlying the UNKNOWN state, but either state also yields the same result, a definitive TRUE results in all three cases.

In database applications

The database structural query language SQL implements ternary logic as a means of handling NULL field content. SQL uses NULL to represent missing data in a database. If a field contains no defined value, SQL assumes this means that an actual value exists, but that value is not currently recorded in the database. Note that a missing value is not the same as either a numeric value of zero, or a string value of zero length. Comparing anything to NULL—even another NULL—results in an UNKNOWN truth state. For example, the SQL expression "City = 'Paris'" resolves to FALSE for a record with "Chicago" in the City field, but it resolves to UNKNOWN for a record with a NULL City field. In other words, to SQL, an undefined field represents potentially any possible value: a missing city *might or might not* represent Paris.

Using ternary logic, SQL can then account for the UNKNOWN truth state in evaluating boolean expressions. Consider the expression "City = 'Paris' OR Balance < 0.0". This expression resolves to TRUE for any record whose Balance field contains a negative number. Likewise, this expression is TRUE for any record with 'Paris' in its City field. The expression resolves to FALSE only for a record whose City field explicitly contains a string other than 'Paris', *and* whose Balance field explicitly contains a non-negative number. In any other case, the expression resolves to UNKNOWN. This is because a missing City value *might be* missing the string 'Paris', and a missing Balance *might be* missing a negative number. However, regardless of missing data, a boolean OR operation is FALSE only when *both* of its operands are also FALSE, so not all missing data leads to an UNKNOWN resolution.

In SQL Data Manipulation Language, a truth state of TRUE for an expression (e.g., in a WHERE clause) initiates an action on a row (e.g. return the row), while a truth state of UNKNOWN or FALSE does not.[4] In this way, ternary logic is implemented in SQL, while behaving as binary logic to the SQL user.

SQL Check Constraints behave differently, however. Only a truth state of FALSE results in a violation of a check constraint. A truth state of TRUE or UNKNOWN indicates a row has been successfully validated against the check constraint[5] .

Electronics

Digital electronics theory supports **four** distinct logic values (as defined in VHDL's **std_logic**):

- **1** or High, usually representing TRUE.
- **0** or Low, usually representing FALSE.
- **X** representing a "Conflict".
- **U** representing "Unassigned" or "Unknown".
- **-** representing "Don't Care".
- **Z** representing "high impedance", undriven line.
- **H**, **L** and **W** are other high-impedance values, the weak pull to "High", "Low" and "Don't Know" correspondingly.

The "X" value does not exist in real-world circuits, it is merely a placeholder used in simulators and for design purposes. Some simulators support representation of the "Z" value, others do not. The "Z" value does exist in real-world circuits but only as an output state.

Use of "X" value in simulation

Many hardware description language (HDL) simulation tools, such as Verilog and VHDL, support an unknown value like that shown above during simulation of digital electronics. The unknown value may be the result of a design error, which the designer can correct before synthesis into an actual circuit. The unknown also represents uninitialised memory values and circuit inputs before the simulation has asserted what the real input value should be.

HDL synthesis tools usually produce circuits that operate only on binary logic.

Use of "X" value in digital design

When designing a digital circuit, some conditions may be outside the scope of the purpose that the circuit will perform. Thus, the designer does not care what happens under those conditions. In addition, the situation occurs that inputs to a circuit are masked by other signals so the value of that input has no effect on circuit behaviour.

In these situations, it is traditional to use "X" as a placeholder to indicate "Don't Care" when building truth tables. This is especially common in state machine design and Karnaugh map simplification. The "X" values provide additional degrees of freedom to the final circuit design, generally resulting in a simplified and smaller circuit.[6]

Once the circuit design is complete and a real circuit is constructed, the "X" values will no longer exist. They will become some tangible "0" or "1" value but could be either depending on the final design optimisation.

Use of "Z" value for high impedance

Some digital devices support a form of three-state logic on their outputs only. The three states are "0", "1", and "Z".

Commonly referred to as **tristate** [7] logic (a trademark of National Semiconductor), it comprises the usual true and false states, with a third *transparent* high impedance state (or 'off-state') which effectively disconnects the logic output. This provides an effective way to connect several logic outputs to a single input, where all but one are put into the high impedance state, allowing the remaining output to operate in the normal binary sense. This is commonly used to connect banks of computer memory and other similar devices to a common data bus; a large number of devices can communicate over the same channel simply by ensuring only one is enabled at a time.

It is important to note that while outputs can have one of three states, inputs can only recognise two. Hence the kind of relations shown in the table above do not occur. Although it could be argued that the high-impedance state is effectively an "unknown", there is absolutely no provision in the vast majority of normal electronics to interpret a high-impedance state as a state in itself. Inputs can only detect "0" and "1".

When a digital input is left disconnected (i.e., when it is given a high impedance signal), the digital value interpreted by the input depends on the type of technology used. TTL technology will reliably default to a "1" state. On the other hand CMOS technology will temporarily hold the previous state seen on that input (due to the capacitance of the gate input). Over time, leakage current causes the CMOS input to drift in a random direction, possibly causing the input state to flip. Disconnected inputs on CMOS devices can pick up noise, they can cause oscillation, the supply current may dramatically increase (crowbar power) or the device may completely destroy itself.

Exotic ternary-logic devices

True ternary logic can be implemented in electronics, although the complexity of design has thus far made it uneconomical to pursue commercially and interest has been primarily confined to research, since 'normal' binary logic is much cheaper to implement and in most cases can easily be configured to emulate ternary systems. However, there are useful applications in fuzzy logic and error correction, and several true ternary logic devices have been manufactured (see external links).

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External links

- Jeff's Trinary Wiki (archived) (http://web.archive.org/web/20080525122206/http://jeff.tk/wiki/Trinary)
- Steve Grubb's Trinary Website (archived) (http://web.archive.org/web/20080611055612/http://www.trinary.cc/)
- Boost.Tribool (http://www.boost.org/doc/html/tribool.html) – an implementation of ternary logic in C++
- Team-R2D2 (http://www.inria.fr/rapportsactivite/RA2004/r2d22004/uid51.html) - a French institute that fabricated the first full-ternary logic chip (a 64-tert SRAM and 4-tert adder) in 2004
- A polar place value number system for computers and life in general (http://www.abhijit.info/tristate/tristate.html)
- Applet on Ternary Char Representation (http://tvm.manojky.net)

Multi-valued logic

Multi-valued logics are 'logical calculi' in which there are more than two truth values. Traditionally, in Aristotle's logical calculus, there were only two possible values (i.e., "true" and "false") for any proposition. An obvious extension to classical two-valued logic is an n-valued logic for $n > 2$. Those most popular in the literature are three-valued (e.g., Łukasiewicz's and Kleene's),—which accept the values "true", "false", and "unknown",—the finite-valued with more than 3 values, and the infinite-valued (e.g. fuzzy logic) logics.

Relation to classical logic

Logics are usually systems intended to codify rules for preserving some semantic property of propositions across transformations. In classical logic, this property is "truth." In a valid argument, the truth of the derived proposition is guaranteed if the premises are jointly true, because the application of valid steps preserves the property. However, that property doesn't have to be that of "truth"; instead, it can be some other concept.

Multi-valued logics are intended to preserve the property of designationhood (or being designated). Since there are more than two truth values, rules of inference may be intended to preserve more than just whichever corresponds (in the relevant sense) to truth. For example, in a three-valued logic, sometimes the two greatest truth-values (when they are represented as e.g. positive integers) are designated and the rules of inference preserve these values. Precisely, a valid argument will be such that the value of the premises taken jointly will always be less than or equal to the conclusion.

For example, the preserved property could be *justification*, the foundational concept of intuitionistic logic. Thus, a proposition is not true or false; instead, it is justified or flawed. A key difference between justification and truth, in this case, is that the law of excluded middle doesn't hold: a proposition that is not flawed is not necessarily justified; instead, it's only not proven that it's flawed. The key difference is the determinacy of the preserved property: One may prove that *P* is justified, that *P* is flawed, or be unable to prove either. A valid argument preserves justification across transformations, so a proposition derived from justified propositions is still justified. However, there are proofs in classical logic that depend upon the law of excluded middle; since that law is not usable under this scheme, there are propositions that cannot be proven that way.

Relation to fuzzy logic

Multi-valued logic is strictly related with fuzzy set theory and fuzzy logic. The notion of fuzzy subset was introduced by Lotfi Zadeh as a formalization of vagueness; i.e., the phenomenon that a predicate may apply to an object not absolutely, but to a certain degree, and that there may be borderline cases. Indeed, as in multi-valued logic, fuzzy logic admits truth values different from "true" and "false". As an example, usually the set of possible truth values is the whole interval [0,1]. Nevertheless, the main difference between fuzzy logic and multi-valued logic is in the aims. In fact, in spite of its philosophical interest (it can be used to deal with the Sorites paradox), fuzzy logic is devoted mainly to the applications. More precisely, there are two approaches to fuzzy logic. The first one is very closely linked with multi-valued logic tradition (Hajek school). So a set of designed values is fixed and this enables us to define an entailment relation. The deduction apparatus is defined by a suitable set of logical axioms and suitable inference rules. Another approach (Goguen, Pavelka and others) is devoted to defining a deduction apparatus in which *approximate reasonings* are admitted. Such an apparatus is defined by a suitable fuzzy subset of logical axioms and by a suitable set of fuzzy inference rules. In the first case the logical consequence operator gives the set of logical consequence of a given set of axioms. In the latter the logical consequence operator gives the fuzzy subset of logical consequence of a given fuzzy subset of hypotheses.

Another example of an infinitely-valued logic is probability logic.

History

The first known classical logician who didn't fully accept the law of excluded middle was Aristotle (who, ironically, is also generally considered to be the first classical logician and the "father of logic"[1]), who admitted that his laws did not all apply to future events (*De Interpretatione*, *ch. IX*). But he didn't create a system of multi-valued logic to explain this isolated remark. The later logicians until the coming of the 20th century followed Aristotelian logic, which includes or assumes the law of the excluded middle.

The 20th century brought the idea of multi-valued logic back. The Polish logician and philosopher Jan Łukasiewicz began to create systems of many-valued logic in 1920, using a third value "possible" to deal with Aristotle's paradox of the sea battle. Meanwhile, the American mathematician Emil L. Post (1921) also introduced the formulation of additional truth degrees with $n \geq 2$,where n are the truth values. Later Jan Łukasiewicz and Alfred Tarski together formulated a logic on n truth values where $n \geq 2$ and in 1932 Hans Reichenbach formulated a logic of many truth values where $n \to$infinity. Kurt Gödel in 1932 showed that intuitionistic logic is not a finitely-many valued logic, and defined a system of Gödel logics intermediate between classical and intuitionistic logic; such logics are known as intermediate logics.

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External links

- Stanford Encyclopedia of Philosophy: "Many-Valued Logic [2]" -- by Siegfried Gottwald.

Notes

[1] Hurley, Patrick. *A Concise Introduction to Logic*, 9th edition. (2006).

[2] http://plato.stanford.edu/entries/logic-manyvalued/

Mathematical logic

Mathematical logic (also known as **symbolic logic**) is a subfield of mathematics with close connections to computer science and philosophical logic.[1] The field includes both the mathematical study of logic and the applications of formal logic to other areas of mathematics. The unifying themes in mathematical logic include the study of the expressive power of formal systems and the deductive power of formal proof systems.

Mathematical logic is often divided into the fields of set theory, model theory, recursion theory, and proof theory. These areas share basic results on logic, particularly first-order logic, and definability. In computer science (particularly in the ACM Classification) mathematical logic encompasses additional topics not detailed in this article; see logic in computer science for those.

Since its inception, mathematical logic has contributed to, and has been motivated by, the study of foundations of mathematics. This study began in the late 19th century with the development of axiomatic frameworks for geometry, arithmetic, and analysis. In the early 20th century it was shaped by David Hilbert's program to prove the consistency of foundational theories. Results of Kurt Gödel, Gerhard Gentzen, and others provided partial resolution to the program, and clarified the issues involved in proving consistency. Work in set theory showed that almost all ordinary mathematics can be formalized in terms of sets, although there are some theorems that cannot be proven in common axiom systems for set theory. Contemporary work in the foundations of mathematics often focuses on establishing which parts of mathematics can be formalized in particular formal systems, rather than trying to find theories in which all of mathematics can be developed.

History

Mathematical logic emerged in the mid-19th century as a subfield of mathematics independent of the traditional study of logic (Ferreirós 2001, p. 443). Before this emergence, logic was studied with rhetoric, through the syllogism, and with philosophy. The first half of the 20th century saw an explosion of fundamental results, accompanied by vigorous debate over the foundations of mathematics.

Early history

Sophisticated theories of logic were developed in many cultures, including China, India, Greece and the Islamic world. In the 18th century, attempts to treat the operations of formal logic in a symbolic or algebraic way had been made by philosophical mathematicians including Leibniz and Lambert, but their labors remained isolated and little known.

19th century

In the middle of the nineteenth century, George Boole and then Augustus De Morgan presented systematic mathematical treatments of logic. Their work, building on work by algebraists such as George Peacock, extended the traditional Aristotelian doctrine of logic into a sufficient framework for the study of foundations of mathematics (Katz 1998, p. 686).

Charles Sanders Peirce built upon the work of Boole to develop a logical system for relations and quantifiers, which he published in several papers from 1870 to 1885. Gottlob Frege presented an independent development of logic with quantifiers in his *Begriffsschrift*, published in 1879, a work generally considered as marking a turning point in the history of logic. Frege's work remained obscure, however, until Bertrand Russell began to promote it near the turn of the century. The two-dimensional notation Frege developed was never widely adopted and is unused in contemporary texts.

From 1890 to 1905, Ernst Schröder published *Vorlesungen über die Algebra der Logik* in three volumes. This work summarized and extended the work of Boole, De Morgan, and Peirce, and was a comprehensive reference to symbolic logic as it was understood at the end of the 19th century.

Foundational theories

Some concerns that mathematics had not been built on a proper foundation led to the development of axiomatic systems for fundamental areas of mathematics such as arithmetic, analysis, and geometry.

In logic, the term *arithmetic* refers to the theory of the natural numbers. Giuseppe Peano (1888) published a set of axioms for arithmetic that came to bear his name (Peano axioms), using a variation of the logical system of Boole and Schröder but adding quantifiers. Peano was unaware of Frege's work at the time. Around the same time Richard Dedekind showed that the natural numbers are uniquely characterized by their induction properties. Dedekind (1888) proposed a different characterization, which lacked the formal logical character of Peano's axioms. Dedekind's work, however, proved theorems inaccessible in Peano's system, including the uniqueness of the set of natural numbers (up to isomorphism) and the recursive definitions of addition and multiplication from the successor function and mathematical induction.

In the mid-19th century, flaws in Euclid's axioms for geometry became known (Katz 1998, p. 774). In addition to the independence of the parallel postulate, established by Nikolai Lobachevsky in 1826 (Lobachevsky 1840), mathematicians discovered that certain theorems taken for granted by Euclid were not in fact provable from his axioms. Among these is the theorem that a line contains at least two points, or that circles of the same radius whose centers are separated by that radius must intersect. Hilbert (1899) developed a complete set of axioms for geometry, building on previous work by Pasch (1882). The success in axiomatizing geometry motivated Hilbert to seek complete axiomatizations of other areas of mathematics, such as the natural numbers and the real line. This would prove to be a major area of research in the first half of the 20th century.

The 19th century saw great advances in the theory of real analysis, including theories of convergence of functions and Fourier series. Mathematicians such as Karl Weierstrass began to construct functions that stretched intuition, such as nowhere-differentiable continuous functions. Previous conceptions of a function as a rule for computation, or a smooth graph, were no longer adequate. Weierstrass began to advocate the arithmetization of analysis, which sought to axiomatize analysis using properties of the natural numbers. The modern (ε, δ)-definition of limit and continuous functions was already developed by Bolzano in 1817 (Felscher 2000), but remained relatively unknown. Cauchy in 1821 defined continuity in terms of infinitesimals (see Cours d'Analyse, page 34). In 1858, Dedekind proposed a definition of the real numbers in terms of Dedekind cuts of rational numbers (Dedekind 1872), a definition still employed in contemporary texts.

Georg Cantor developed the fundamental concepts of infinite set theory. His early results developed the theory of cardinality and proved that the reals and the natural numbers have different cardinalities (Cantor 1874). Over the next twenty years, Cantor developed a theory of transfinite numbers in a series of publications. In 1891, he published a new proof of the uncountability of the real numbers that introduced the diagonal argument, and used this method to prove Cantor's theorem that no set can have the same cardinality as its powerset. Cantor believed that every set could be well-ordered, but was unable to produce a proof for this result, leaving it as an open problem in 1895 (Katz 1998, p. 807).

20th century

In the early decades of the 20th century, the main areas of study were set theory and formal logic. The discovery of paradoxes in informal set theory caused some to wonder whether mathematics itself is inconsistent, and to look for proofs of consistency.

In 1900, Hilbert posed a famous list of 23 problems for the next century. The first two of these were to resolve the continuum hypothesis and prove the consistency of elementary arithmetic, respectively; the tenth was to produce a method that could decide whether a multivariate polynomial equation over the integers has a solution. Subsequent work to resolve these problems shaped the direction of mathematical logic, as did the effort to resolve Hilbert's *Entscheidungsproblem*, posed in 1928. This problem asked for a procedure that would decide, given a formalized mathematical statement, whether the statement is true or false.

Set theory and paradoxes

Ernst Zermelo (1904) gave a proof that every set could be well-ordered, a result Georg Cantor had been unable to obtain. To achieve the proof, Zermelo introduced the axiom of choice, which drew heated debate and research among mathematicians and the pioneers of set theory. The immediate criticism of the method led Zermelo to publish a second exposition of his result, directly addressing criticisms of his proof (Zermelo 1908a). This paper led to the general acceptance of the axiom of choice in the mathematics community.

Skepticism about the axiom of choice was reinforced by recently discovered paradoxes in naive set theory. Cesare Burali-Forti (1897) was the first to state a paradox: the Burali-Forti paradox shows that the collection of all ordinal numbers cannot form a set. Very soon thereafter, Bertrand Russell discovered Russell's paradox in 1901, and Jules Richard (1905) discovered Richard's paradox.

Zermelo (1908b) provided the first set of axioms for set theory. These axioms, together with the additional axiom of replacement proposed by Abraham Fraenkel, are now called Zermelo–Fraenkel set theory (ZF). Zermelo's axioms incorporated the principle of limitation of size to avoid Russell's paradox.

In 1910, the first volume of *Principia Mathematica* by Russell and Alfred North Whitehead was published. This seminal work developed the theory of functions and cardinality in a completely formal framework of type theory, which Russell and Whitehead developed in an effort to avoid the paradoxes. *Principia Mathematica* is considered one of the most influential works of the 20th century, although the framework of type theory did not prove popular as a foundational theory for mathematics (Ferreirós 2001, p. 445).

Fraenkel (1922) proved that the axiom of choice cannot be proved from the remaining axioms of Zermelo's set theory with urelements. Later work by Paul Cohen (1966) showed that the addition of urelements is not needed, and the axiom of choice is unprovable in ZF. Cohen's proof developed the method of forcing, which is now an important tool for establishing independence results in set theory.

Symbolic logic

Leopold Löwenheim (1915) and Thoralf Skolem (1920) obtained the Löwenheim–Skolem theorem, which says that first-order logic cannot control the cardinalities of infinite structures. Skolem realized that this theorem would apply to first-order formalizations of set theory, and that it implies any such formalization has a countable model. This counterintuitive fact became known as Skolem's paradox.

In his doctoral thesis, Kurt Gödel (1929) proved the completeness theorem, which establishes a correspondence between syntax and semantics in first-order logic. Gödel used the completeness theorem to prove the compactness theorem, demonstrating the finitary nature of first-order logical consequence. These results helped establish first-order logic as the dominant logic used by mathematicians.

In 1931, Gödel published *On Formally Undecidable Propositions of Principia Mathematica and Related Systems*, which proved the incompleteness (in a different meaning of the word) of all sufficiently strong, effective first-order theories. This result, known as Gödel's incompleteness theorem, establishes severe limitations on axiomatic foundations for mathematics, striking a strong blow to Hilbert's program. It showed the impossibility of providing a consistency proof of arithmetic within any formal theory of arithmetic. Hilbert, however, did not acknowledge the importance of the incompleteness theorem for some time.

Gödel's theorem shows that a consistency proof of any sufficiently strong, effective axiom system cannot be obtained in the system itself, if the system is consistent, nor in any weaker system. This leaves open the possibility of consistency proofs that cannot be formalized within the system they consider. Gentzen (1936) proved the consistency of arithmetic using a finitistic system together with a principle of transfinite induction. Gentzen's result introduced the ideas of cut elimination and proof-theoretic ordinals, which became key tools in proof theory. Gödel (1958) gave a different consistency proof, which reduces the consistency of classical arithmetic to that of intutitionistic arithmetic in higher types.

Beginnings of the other branches

Alfred Tarski developed the basics of model theory.

Beginning in 1935, a group of prominent mathematicians collaborated under the pseudonym Nicolas Bourbaki to publish a series of encyclopedic mathematics texts. These texts, written in an austere and axiomatic style, emphasized rigorous presentation and set-theoretic foundations. Terminology coined by these texts, such as the words *bijection*, *injection*, and *surjection*, and the set-theoretic foundations the texts employed, were widely adopted throughout mathematics.

The study of computability came to be known as recursion theory, because early formalizations by Gödel and Kleene relied on recursive definitions of functions.[2] When these definitions were shown equivalent to Turing's formalization involving Turing machines, it became clear that a new concept – the computable function – had been discovered, and that this definition was robust enough to admit numerous independent characterizations. In his work on the incompleteness theorems in 1931, Gödel lacked a rigorous concept of an effective formal system; he immediately realized that the new definitions of computability could be used for this purpose, allowing him to state the incompleteness theorems in generality that could only be implied in the original paper.

Numerous results in recursion theory were obtained in the 1940s by Stephen Cole Kleene and Emil Leon Post. Kleene (1943) introduced the concepts of relative computability, foreshadowed by Turing (1939), and the arithmetical hierarchy. Kleene later generalized recursion theory to higher-order functionals. Kleene and Kreisel studied formal versions of intuitionistic mathematics, particularly in the context of proof theory.

Subfields and scope

The *Handbook of Mathematical Logic* makes a rough division of contemporary mathematical logic into four areas:

1. set theory
2. model theory
3. recursion theory, and
4. proof theory and constructive mathematics (considered as parts of a single area).

Each area has a distinct focus, although many techniques and results are shared between multiple areas. The border lines between these fields, and the lines between mathematical logic and other fields of mathematics, are not always sharp. Gödel's incompleteness theorem marks not only a milestone in recursion theory and proof theory, but has also led to Löb's theorem in modal logic. The method of forcing is employed in set theory, model theory, and recursion theory, as well as in the study of intuitionistic mathematics.

The mathematical field of category theory uses many formal axiomatic methods, and includes the study of categorical logic, but category theory is not ordinarily considered a subfield of mathematical logic. Because of its applicability in diverse fields of mathematics, mathematicians including Saunders Mac Lane have proposed category theory as a foundational system for mathematics, independent of set theory. These foundations use toposes, which resemble generalized models of set theory that may employ classical or nonclassical logic.

Formal logical systems

At its core, mathematical logic deals with mathematical concepts expressed using formal logical systems. These systems, though they differ in many details, share the common property of considering only expressions in a fixed formal language, or signature. The system of first-order logic is the most widely studied today, because of its applicability to foundations of mathematics and because of its desirable proof-theoretic properties.[3] Stronger classical logics such as second-order logic or infinitary logic are also studied, along with nonclassical logics such as intuitionistic logic.

First-order logic

First-order logic is a particular formal system of logic. Its syntax involves only finite expressions as well-formed formulas, while its semantics are characterized by the limitation of all quantifiers to a fixed domain of discourse.

Early results about formal logic established limitations of first-order logic. The Löwenheim–Skolem theorem (1919) showed that if a set of sentences in a countable first-order language has an infinite model then it has at least one model of each infinite cardinality. This shows that it is impossible for a set of first-order axioms to characterize the natural numbers, the real numbers, or any other infinite structure up to isomorphism. As the goal of early foundational studies was to produce axiomatic theories for all parts of mathematics, this limitation was particularly stark.

Gödel's completeness theorem (Gödel 1929) established the equivalence between semantic and syntactic definitions of logical consequence in first-order logic. It shows that if a particular sentence is true in every model that satisfies a particular set of axioms, then there must be a finite deduction of the sentence from the axioms. The compactness theorem first appeared as a lemma in Gödel's proof of the completeness theorem, and it took many years before logicians grasped its significance and began to apply it routinely. It says that a set of sentences has a model if and only if every finite subset has a model, or in other words that an inconsistent set of formulas must have a finite inconsistent subset. The completeness and compactness theorems allow for sophisticated analysis of logical consequence in first-order logic and the development of model theory, and they are a key reason for the prominence of first-order logic in mathematics.

Gödel's incompleteness theorems (Gödel 1931) establish additional limits on first-order axiomatizations. The **first incompleteness theorem** states that for any sufficiently strong, effectively given logical system there exists a statement which is true but not provable within that system. Here a logical system is effectively given if it is possible to decide, given any formula in the language of the system, whether the formula is an axiom. A logical system is sufficiently strong if it can express the Peano axioms. When applied to first-order logic, the first incompleteness theorem implies that any sufficiently strong, consistent, effective first-order theory has models that are not elementarily equivalent, a stronger limitation than the one established by the Löwenheim–Skolem theorem. The **second incompleteness theorem** states that no sufficiently strong, consistent, effective axiom system for arithmetic can prove its own consistency, which has been interpreted to show that Hilbert's program cannot be completed.

Other classical logics

Many logics besides first-order logic are studied. These include infinitary logics, which allow for formulas to provide an infinite amount of information, and higher-order logics, which include a portion of set theory directly in their semantics.

The most well studied infinitary logic is $L_{\omega_1,\omega}$. In this logic, quantifiers may only be nested to finite depths, as in first order logic, but formulas may have finite or countably infinite conjunctions and disjunctions within them. Thus, for example, it is possible to say that an object is a whole number using a formula of $L_{\omega_1,\omega}$ such as

$$(x = 0) \vee (x = 1) \vee (x = 2) \vee \cdots.$$

Higher-order logics allow for quantification not only of elements of the domain of discourse, but subsets of the domain of discourse, sets of such subsets, and other objects of higher type. The semantics are defined so that, rather than having a separate domain for each higher-type quantifier to range over, the quantifiers instead range over all objects of the appropriate type. The logics studied before the development of first-order logic, for example Frege's logic, had similar set-theoretic aspects. Although higher-order logics are more expressive, allowing complete axiomatizations of structures such as the natural numbers, they do not satisfy analogues of the completeness and compactness theorems from first-order logic, and are thus less amenable to proof-theoretic analysis.

Another type of logics are fixed-point logics that allow inductive definitions, like one writes for primitive recursive functions.

One can formally define an extension of first-order logic — a notion which encompasses all logics in this section because they behave like first-order logic in certain fundamental ways, but does not encompass all logics in general, e.g. it does not encompass intuitionistic, modal or fuzzy logic. Lindström's theorem implies that the only extension of first-order logic satisfying both the Compactness theorem and the Downward Löwenheim–Skolem theorem is first-order logic.

Nonclassical and modal logic

Modal logics include additional modal operators, such as an operator which states that a particular formula is not only true, but necessarily true. Although modal logic is not often used to axiomatize mathematics, it has been used to study the properties of first-order provability (Solovay 1976) and set-theoretic forcing (Hamkins and Löwe 2007).

Intuitionistic logic was developed by Heyting to study Brouwer's program of intuitionism, in which Brouwer himself avoided formalization. Intuitionistic logic specifically does not include the law of the excluded middle, which states that each sentence is either true or its negation is true. Kleene's work with the proof theory of intuitionistic logic showed that constructive information can be recovered from intuitionistic proofs. For example, any provably total function in intuitionistic arithmetic is computable; this is not true in classical theories of arithmetic such as Peano arithmetic.

Algebraic logic

Algebraic logic uses the methods of abstract algebra to study the semantics of formal logics. A fundamental example is the use of Boolean algebras to represent truth values in classical propositional logic, and the use of Heyting algebras to represent truth values in intuitionistic propositional logic. Stronger logics, such as first-order logic and higher-order logic, are studied using more complicated algebraic structures such as cylindric algebras.

Set theory

Set theory is the study of sets, which are abstract collections of objects. Many of the basic notions, such as ordinal and cardinal numbers, were developed informally by Cantor before formal axiomatizations of set theory were developed. The first such axiomatization, due to Zermelo (1908b), was extended slightly to become Zermelo–Fraenkel set theory (ZF), which is now the most widely used foundational theory for mathematics.

Other formalizations of set theory have been proposed, including von Neumann–Bernays–Gödel set theory (NBG), Morse–Kelley set theory (MK), and New Foundations (NF). Of these, ZF, NBG, and MK are similar in describing a cumulative hierarchy of sets. New Foundations takes a different approach; it allows objects such as the set of all sets at the cost of restrictions on its set-existence axioms. The system of Kripke-Platek set theory is closely related to generalized recursion theory.

Two famous statements in set theory are the axiom of choice and the continuum hypothesis. The axiom of choice, first stated by Zermelo (1904), was proved independent of ZF by Fraenkel (1922), but has come to be widely accepted by mathematicians. It states that given a collection of nonempty sets there is a single set *C* that contains exactly one element from each set in the collection. The set *C* is said to "choose" one element from each set in the collection. While the ability to make such a choice is considered obvious by some, since each set in the collection is nonempty, the lack of a general, concrete rule by which the choice can be made renders the axiom nonconstructive. Stefan Banach and Alfred Tarski (1924) showed that the axiom of choice can be used to decompose a solid ball into a finite number of pieces which can then be rearranged, with no scaling, to make two solid balls of the original size. This theorem, known as the Banach-Tarski paradox, is one of many counterintuitive results of the axiom of choice.

The continuum hypothesis, first proposed as a conjecture by Cantor, was listed by David Hilbert as one of his 23 problems in 1900. Gödel showed that the continuum hypothesis cannot be disproven from the axioms of Zermelo–Fraenkel set theory (with or without the axiom of choice), by developing the constructible universe of set theory in which the continuum hypothesis must hold. In 1963, Paul Cohen showed that the continuum hypothesis cannot be proven from the axioms of Zermelo–Fraenkel set theory (Cohen 1966). This independence result did not completely settle Hilbert's question, however, as it is possible that new axioms for set theory could resolve the hypothesis. Recent work along these lines has been conducted by W. Hugh Woodin, although its importance is not yet clear (Woodin 2001).

Contemporary research in set theory includes the study of large cardinals and determinacy. Large cardinals are cardinal numbers with particular properties so strong that the existence of such cardinals cannot be proved in ZFC. The existence of the smallest large cardinal typically studied, an inaccessible cardinal, already implies the consistency of ZFC. Despite the fact that large cardinals have extremely high cardinality, their existence has many ramifications for the structure of the real line. *Determinacy* refers to the possible existence of winning strategies for certain two-player games (the games are said to be *determined*). The existence of these strategies implies structural properties of the real line and other Polish spaces.

Model theory

Model theory studies the models of various formal theories. Here a theory is a set of formulas in a particular formal logic and signature, while a model is a structure that gives a concrete interpretation of the theory. Model theory is closely related to universal algebra and algebraic geometry, although the methods of model theory focus more on logical considerations than those fields.

The set of all models of a particular theory is called an elementary class; classical model theory seeks to determine the properties of models in a particular elementary class, or determine whether certain classes of structures form elementary classes.

The method of quantifier elimination can be used to show that definable sets in particular theories cannot be too complicated. Tarski (1948) established quantifier elimination for real-closed fields, a result which also shows the theory of the field of real numbers is decidable. (He also noted that his methods were equally applicable to algebraically closed fields of arbitrary characteristic.) A modern subfield developing from this is concerned with o-minimal structures.

Morley's categoricity theorem, proved by Michael D. Morley (1965), states that if a first-order theory in a countable language is categorical in some uncountable cardinality, i.e. all models of this cardinality are isomorphic, then it is categorical in all uncountable cardinalities.

A trivial consequence of the continuum hypothesis is that a complete theory with less than continuum many nonisomorphic countable models can have only countably many. Vaught's conjecture, named after Robert Lawson Vaught, says that this is true even independently of the continuum hypothesis. Many special cases of this conjecture have been established.

Recursion theory

Recursion theory, also called **computability theory**, studies the properties of computable functions and the Turing degrees, which divide the uncomputable functions into sets which have the same level of uncomputability. Recursion theory also includes the study of generalized computability and definability. Recursion theory grew from of the work of Alonzo Church and Alan Turing in the 1930s, which was greatly extended by Kleene and Post in the 1940s.

Classical recursion theory focuses on the computability of functions from the natural numbers to the natural numbers. The fundamental results establish a robust, canonical class of computable functions with numerous independent, equivalent characterizations using Turing machines, λ calculus, and other systems. More advanced results concern the structure of the Turing degrees and the lattice of recursively enumerable sets.

Generalized recursion theory extends the ideas of recursion theory to computations that are no longer necessarily finite. It includes the study of computability in higher types as well as areas such as hyperarithmetical theory and α-recursion theory.

Contemporary research in recursion theory includes the study of applications such as algorithmic randomness, computable model theory, and reverse mathematics, as well as new results in pure recursion theory.

Algorithmically unsolvable problems

An important subfield of recursion theory studies algorithmic unsolvability; a decision problem or function problem is **algorithmically unsolvable** if there is no possible computable algorithm which returns the correct answer for all legal inputs to the problem. The first results about unsolvability, obtained independently by Church and Turing in 1936, showed that the Entscheidungsproblem is algorithmically unsolvable. Turing proved this by establishing the unsolvability of the halting problem, a result with far-ranging implications in both recursion theory and computer science.

There are many known examples of undecidable problems from ordinary mathematics. The word problem for groups was proved algorithmically unsolvable by Pyotr Novikov in 1955 and independently by W. Boone in 1959. The busy beaver problem, developed by Tibor Radó in 1962, is another well-known example.

Hilbert's tenth problem asked for an algorithm to determine whether a multivariate polynomial equation with integer coefficients has a solution in the integers. Partial progress was made by Julia Robinson, Martin Davis and Hilary Putnam. The algorithmic unsolvability of the problem was proved by Yuri Matiyasevich in 1970 (Davis 1973).

Proof theory and constructive mathematics

Proof theory is the study of formal proofs in various logical deduction systems. These proofs are represented as formal mathematical objects, facilitating their analysis by mathematical techniques. Several deduction systems are commonly considered, including Hilbert-style deduction systems, systems of natural deduction, and the sequent calculus developed by Gentzen.

The study of **constructive mathematics**, in the context of mathematical logic, includes the study of systems in non-classical logic such as intuitionistic logic, as well as the study of predicative systems. An early proponent of predicativism was Hermann Weyl, who showed it is possible to develop a large part of real analysis using only predicative methods (Weyl 1918).

Because proofs are entirely finitary, whereas truth in a structure is not, it is common for work in constructive mathematics to emphasize provability. The relationship between provability in classical (or nonconstructive) systems and provability in intuitionistic (or constructive, respectively) systems is of particular interest. Results such as the Gödel–Gentzen negative translation show that it is possible to embed (or *translate*) classical logic into intuitionistic logic, allowing some properties about intuitionistic proofs to be transferred back to classical proofs.

Recent developments in proof theory include the study of proof mining by Ulrich Kohlenbach and the study of proof-theoretic ordinals by Michael Rathjen.

Connections with computer science

The study of computability theory in computer science is closely related to the study of computability in mathematical logic. There is a difference of emphasis, however. Computer scientists often focus on concrete programming languages and feasible computability, while researchers in mathematical logic often focus on computability as a theoretical concept and on noncomputability.

The theory of semantics of programming languages is related to model theory, as is program verification (in particular, model checking). The Curry–Howard isomorphism between proofs and programs relates to proof theory, especially intuitionistic logic. Formal calculi such as the lambda calculus and combinatory logic are now studied as idealized programming languages.

Computer science also contributes to mathematics by developing techniques for the automatic checking or even finding of proofs, such as automated theorem proving and logic programming.

Descriptive complexity theory relates logics to computational complexity. The first significant result in this area, Fagin's theorem (1974) established that NP is precisely the set of languages expressible by sentences of existential second-order logic.

Foundations of mathematics

In the 19th century, mathematicians became aware of logical gaps and inconsistencies in their field. It was shown that Euclid's axioms for geometry, which had been taught for centuries as an example of the axiomatic method, were incomplete. The use of infinitesimals, and the very definition of function, came into question in analysis, as pathological examples such as Weierstrass' nowhere-differentiable continuous function were discovered.

Cantor's study of arbitrary infinite sets also drew criticism. Leopold Kronecker famously stated "God made the integers; all else is the work of man," endorsing a return to the study of finite, concrete objects in mathematics. Although Kronecker's argument was carried forward by constructivists in the 20th century, the mathematical community as a whole rejected them. David Hilbert argued in favor of the study of the infinite, saying "No one shall expel us from the Paradise that Cantor has created."

Mathematicians began to search for axiom systems that could be used to formalize large parts of mathematics. In addition to removing ambiguity from previously-naive terms such as function, it was hoped that this axiomatization would allow for consistency proofs. In the 19th century, the main method of proving the consistency of a set of axioms was to provide a model for it. Thus, for example, non-Euclidean geometry can be proved consistent by defining *point* to mean a point on a fixed sphere and *line* to mean a great circle on the sphere. The resulting structure, a model of elliptic geometry, satisfies the axioms of plane geometry except the parallel postulate.

With the development of formal logic, Hilbert asked whether it would be possible to prove that an axiom system is consistent by analyzing the structure of possible proofs in the system, and showing through this analysis that it is impossible to prove a contradiction. This idea led to the study of proof theory. Moreover, Hilbert proposed that the analysis should be entirely concrete, using the term *finitary* to refer to the methods he would allow but not precisely defining them. This project, known as Hilbert's program, was seriously affected by Gödel's incompleteness theorems, which show that the consistency of formal theories of arithmetic cannot be established using methods formalizable in those theories. Gentzen showed that it is possible to produce a proof of the consistency of arithmetic in a finitary system augmented with axioms of transfinite induction, and the techniques he developed to so do were seminal in proof theory.

A second thread in the history of foundations of mathematics involves nonclassical logics and constructive mathematics. The study of constructive mathematics includes many different programs with various definitions of *constructive*. At the most accommodating end, proofs in ZF set theory that do not use the axiom of choice are called constructive by many mathematicians. More limited versions of constructivism limit themselves to natural numbers, number-theoretic functions, and sets of natural numbers (which can be used to represent real numbers, facilitating the study of mathematical analysis). A common idea is that a concrete means of computing the values of the function must be known before the function itself can be said to exist.

In the early 20th century, Luitzen Egbertus Jan Brouwer founded intuitionism as a philosophy of mathematics. This philosophy, poorly understood at first, stated that in order for a mathematical statement to be true to a mathematician, that person must be able to *intuit* the statement, to not only believe its truth but understand the reason for its truth. A consequence of this definition of truth was the rejection of the law of the excluded middle, for there are statements that, according to Brouwer, could not be claimed to be true while their negations also could not be claimed true. Brouwer's philosophy was influential, and the cause of bitter disputes among prominent mathematicians. Later, Kleene and Kreisel would study formalized versions of intuitionistic logic (Brouwer rejected formalization, and presented his work in unformalized natural language). With the advent of the BHK interpretation and Kripke models, intuitionism became easier to reconcile with classical mathematics.

See also

- List of mathematical logic topics
- List of computability and complexity topics
- List of set theory topics
- List of first-order theories
- Knowledge representation
- Metalogic
- Logic symbols

Notes

[1] Undergraduate texts include Boolos, Burgess, and Jeffrey (2002), Enderton (2001), and Mendelson (1997). A classic graduate text by Shoenfield (2001) first appeared in 1967.

[2] A detailed study of this terminology is given by Soare (1996).

[3] Ferreirós (2001) surveys the rise of first-order logic over other formal logics in the early 20th century.

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- Stanford Encyclopedia of Philosophy: First-order Model Theory (http://plato.stanford.edu/entries/modeltheory-fo/) – by Wilfrid Hodges.
- The London Philosophy Study Guide (http://www.ucl.ac.uk/philosophy/LPSG/) offers many suggestions on what to read, depending on the student's familiarity with the subject:
  - Mathematical Logic (http://www.ucl.ac.uk/philosophy/LPSG/MathLogic.htm)
  - Set Theory & Further Logic (http://www.ucl.ac.uk/philosophy/LPSG/SetTheory.htm)

- Philosophy of Mathematics (http://www.ucl.ac.uk/philosophy/LPSG/PhilMath.htm)

Symbolic logic

Symbolic logic may refer to:

- First-order logic, a system of formal logic
- Mathematical logic, a field of mathematics

Metalogic

Metalogic is the study of the metatheory of logic. While *logic* is the study of the manner in which logical systems can be used to decide the correctness of arguments, metalogic studies the properties of the logical systems themselves.[1] According to Geoffrey Hunter, while logic concerns itself with the "truths of logic," metalogic concerns itself with the theory of "sentences used to express truths of logic."[2]

The basic objects of study in metalogic are formal languages, formal systems, and their interpretations. The study of interpretation of formal systems is the branch of mathematical logic known as model theory, while the study of deductive apparatus is the branch known as proof theory.

History

Metalogical questions have been asked since the time of Aristotle. However, it was only with the rise of formal languages in the late 19th and early 20th century that investigations into the foundations of logic began to flourish. In 1904, David Hilbert observed that in investigating the foundations of mathematics that logical notions are presupposed, and therefore a simultaneous account of metalogical and metamathematical principles was required. Today, metalogic and metamathematics are largely synonymous with each other, and both have been substantially subsumed by mathematical logic in academia.

Important distinctions in metalogic

Metalanguage–Object language

In metalogic, formal languages are sometimes called *object languages*. The language used to make statements about an object language is called a *metalanguage*. This distinction is a key difference between logic and metalogic. While logic deals with *proofs in a formal system*, expressed in some formal language, metalogic deals with *proofs about a formal system* which are expressed in a metalanguage about some object language.

Syntax–semantics

In metalogic, 'syntax' has to do with formal languages or formal systems without regard to any interpretation of them, whereas, 'semantics' has to do with interpretations of formal languages. The term 'syntactic' has a slightly wider scope than 'proof-theoretic', since it may be applied to properties of formal languages without any deductive systems, as well as to formal systems. 'Semantic' is synonymous with 'model-theoretic'.

Use–mention

In metalogic, the words 'use' and 'mention', in both their noun and verb forms, take on a technical sense in order to identify an important distinction.[2] The *use–mention distinction* (sometimes referred to as the *words-as-words distinction*) is the distinction between *using* a word (or phrase) and *mentioning* it. Usually it is indicated that an expression is being mentioned rather than used by enclosing it in quotation marks, printing it in italics, or setting the expression by itself on a line. The enclosing in quotes of an expression gives us the name of an expression, for example:

> 'Metalogic' is the name of this article.
>
> This article is about metalogic.

Type–token

The *type-token distinction* is a distinction in metalogic, that separates an abstract concept from the objects which are particular instances of the concept. For example, the particular bicycle in your garage is a token of the type of thing known as "The bicycle." Whereas, the bicycle in your garage is in a particular place at a particular time, that is not true of "the bicycle" as used in the sentence: "*The bicycle* has become more popular recently." This distinction is used to clarify the meaning of symbols of formal languages.

Overview

Formal language

A *formal language* is an organized set of symbols the essential feature of which is that it can be precisely defined in terms of just the shapes and locations of those symbols. Such a language can be defined, then, without any reference to any meanings of any of its expressions; it can exist before any interpretation is assigned to it—that is, before it has any meaning. First order logic is expressed in some formal language. A formal grammar determines which symbols and sets of symbols are formulas in a formal language.

A formal language can be defined formally as a set *A* of strings (finite sequences) on a fixed alphabet α. Some authors, including Carnap, define the language as the ordered pair <α, *A*>.[3] Carnap also requires that each element of α must occur in at least one string in *A*.

Formation rules

Formation rules (also called *formal grammar*) are a precise description of the well-formed formulas of a formal language. It is synonymous with the set of strings over the alphabet of the formal language which constitute well formed formulas. However, it does not describe their semantics (i.e. what they mean).

Formal systems

A *formal system* (also called a *logical calculus*, or a *logical system*) consists of a formal language together with a deductive apparatus (also called a *deductive system*). The deductive apparatus may consist of a set of transformation rules (also called *inference rules*) or a set of axioms, or have both. A formal system is used to derive one expression from one or more other expressions.

A *formal system* can be formally defined as an ordered triple $<\alpha, \mathcal{I}, \mathcal{D}d>$, where $\mathcal{D}d$ is the relation of direct derivability. This relation is understood in a comprehensive sense such that the primitive sentences of the formal system are taken as directly derivable from the empty set of sentences. Direct derivability is a relation between a sentence and a finite, possibly empty set of sentences. Axioms are laid down in such a way that every first place member of $\mathcal{D}d$ is a member of $\mathcal{I}$ and every second place member is a finite subset of $\mathcal{I}$.

It is also possible to define a *formal system* using only the relation $\mathcal{D}$ d. In this way we can omit $\mathcal{I}$, and α in the definitions of *interpreted formal language*, and *interpreted formal system*. However, this method can be more difficult to understand and work with.[3]

Formal proofs

A *formal proof* is a sequence of well-formed formulas of a formal language, the last one of which is a theorem of a formal system. The theorem is a syntactic consequence of all the well formed formulae preceding it in the proof. For a well formed formula to qualify as part of a proof, it must be the result of applying a rule of the deductive apparatus of some formal system to the previous well formed formulae in the proof sequence.

Interpretations

An *interpretation* of a formal system is the assignment of meanings, to the symbols, and truth-values to the sentences of the formal system. The study of interpretations is called Formal semantics. *Giving an interpretation* is synonymous with *constructing a model.*

Results in metalogic

Results in metalogic consist of such things as formal proofs demonstrating the consistency, completeness, and decidability of particular formal systems.

Major results in metalogic include:

- Proof of the uncountability of the set of all subsets of the set of natural numbers (Cantor's theorem 1891)
- Löwenheim-Skolem theorem (Leopold Löwenheim 1915 and Thoralf Skolem 1919)
- Proof of the consistency of truth-functional propositional logic (Emil Post 1920)
- Proof of the semantic completeness of truth-functional propositional logic (Paul Bernays 1918),[4] (Emil Post 1920)[2]
- Proof of the syntactic completeness of truth-functional propositional logic (Emil Post 1920)[2]
- Proof of the decidability of truth-functional propositional logic (Emil Post 1920)[2]
- Proof of the consistency of first order monadic predicate logic (Leopold Löwenheim 1915)
- Proof of the semantic completeness of first order monadic predicate logic (Leopold Löwenheim 1915)
- Proof of the decidability of first order monadic predicate logic (Leopold Löwenheim 1915)
- Proof of the consistency of first order predicate logic (David Hilbert and Wilhelm Ackermann 1928)
- Proof of the semantic completeness of first order predicate logic (Gödel's completeness theorem 1930)
- Proof of the undecidability of first order predicate logic (Church's theorem 1936)
- Gödel's first incompleteness theorem 1931
- Gödel's second incompleteness theorem 1931

See also

- Metamathematics

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[2] Hunter, Geoffrey, *Metalogic: An Introduction to the Metatheory of Standard First-Order Logic*, University of California Press, 1971
[3] Rudolf Carnap (1958) *Introduction to Symbolic Logic and its Applications*, p. 102.
[4] Hao Wang, Reflections on Kurt Gödel

Metatheory

A **metatheory** or **meta-theory** is a theory whose subject matter is some other theory. In other words it is a theory about a theory. Statements made in the metatheory about the theory are called metatheorems.

According to the systemic TOGA meta-theory [1] , a meta-theory may refer to the specific point of view on a theory and to its subjective meta-properties, but not to its application domain. In the above sense, a theory **T** of the domain **D** is a meta-theory if **D** is a theory or a set of theories. A general theory is not a meta-theory because its domain **D** are not theories.

The following is an example of a meta-theoretical statement:[2]

> Any physical theory is always provisional, in the sense that it is only a hypothesis; you can never prove it. No matter how many times the results of experiments agree with some theory, you can never be sure that the next time the result will not contradict the theory. On the other hand, you can disprove a theory by finding even a single observation that disagrees with the predictions of the theory.

Meta-theory belongs to the philosophical specialty of epistemology and metamathematics, as well as being an object of concern to the area in which the individual theory is conceived. An emerging domain of meta-theories is systemics.

Taxonomy

Examining groups of related theories, a first finding may be to identify classes of theories, thus specifying a taxonomy of theories. A proof engendered by a metatheory is called a *metatheorem*.

History

The concept burst upon the scene of twentieth-century philosophy as a result of the work of the German mathematician David Hilbert, who in 1905 published a proposal for proof of the consistency of mathematics, creating the field of metamathematics. His hopes for the success of this proof were dashed by the work of Kurt Gödel who in 1931 proved this to be unattainable by his incompleteness theorems. Nevertheless, his program of unsolved mathematical problems, out of which grew this metamathematical proposal, continued to influence the direction of mathematics for the rest of the twentieth century.

The study of metatheory became widespread during the rest of that century by its application in other fields, notably scientific linguistics and its concept of metalanguage.

See also

- meta-
- meta-knowledge
- Metalogic
- Metamathematics
- Metahistory, a book by Hayden White
- Philosophy of social science

References

[1] * Meta-Knowledge Unified Framework (http://hid.casaccia.enea.it/Meta-know-1.htm) - the TOGA meta-theory
[2] Stephen Hawking in *A Brief History of Time*

External links

- Meta-theoretical Issues (2003), Lyle Flint (http://www.bsu.edu/classes/flint/comm360/metatheo.html)

Metamathematics

Metamathematics is the study of mathematics itself using mathematical methods. This study produces metatheories, which are mathematical theories about other mathematical theories. Metamathematical metatheorems about mathematics itself were originally differentiated from ordinary mathematical theorems in the 19th century, to focus on what was then called the foundational crisis of mathematics. Richard's paradox (Richard 1905) concerning certain 'definitions' of real numbers in the English language is an example of the sort of contradictions which can easily occur if one fails to distinguish between mathematics and metamathematics.

The term "metamathematics" is sometimes used as a synonym for certain elementary parts of formal logic, including propositional logic and predicate logic.

History

Metamathematics was intimately connected to mathematical logic, so that the early histories of the two fields, during the late 19th and early 20th centuries, largely overlap. More recently, mathematical logic has often included the study of new pure mathematics, such as set theory, recursion theory and pure model theory, which is not directly related to metamathematics.

Serious metamathematical reflection began with the work of Gottlob Frege, especially his *Begriffsschrift*.

David Hilbert was the first to invoke the term "metamathematics" with regularity (see Hilbert's program). In his hands, it meant something akin to contemporary proof theory, in which finitary methods are used to study various axiomatized mathematical theorems.

Other prominent figures in the field include Bertrand Russell, Thoralf Skolem, Emil Post, Alonzo Church, Stephen Kleene, Willard Quine, Paul Benacerraf, Hilary Putnam, Gregory Chaitin, Alfred Tarski and Kurt Gödel. In particular, arguably the greatest achievement of metamathematics and the philosophy of mathematics to date is Gödel's incompleteness theorem: proof that given any finite number of axioms for Peano arithmetic, there will be true statements about that arithmetic that cannot be proved from those axioms.

Milestones

- Principia Mathematica (Whitehead and Russell 1925)
- Gödel's completeness theorem, 1930
- Gödel's incompleteness theorem, 1931
- Tarski's definition of model-theoretic satisfaction, now called the T-schema
- The proof of the impossibility of the Entscheidungsproblem, obtained independently in 1936–1937 by Church and Turing.

See also

- Meta-
- Model theory
- Philosophy of mathematics
- Proof theory

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Abstract Algebra

Abstract algebra

Abstract algebra is the subject area of mathematics that studies algebraic structures, such as groups, rings, fields, modules, vector spaces, and algebras. The phrase **abstract algebra** was coined at the turn of the 20th century to distinguish this area from what was normally referred to as **algebra**, the study of the rules for manipulating formulae and algebraic expressions involving unknowns and real or complex numbers, often now called *elementary algebra*. The distinction is rarely made in more recent writings.

Contemporary mathematics and mathematical physics make extensive use of abstract algebra; for example, theoretical physics draws on Lie algebras. Subject areas such as algebraic number theory, algebraic topology, and algebraic geometry apply algebraic methods to other areas of mathematics. Representation theory, roughly speaking, takes the 'abstract' out of 'abstract algebra', studying the concrete side of a given structure; see model theory.

Two mathematical subject areas that study the properties of algebraic structures viewed as a whole are universal algebra and category theory. Algebraic structures, together with the associated homomorphisms, form categories. Category theory is a powerful formalism for studying and comparing different algebraic structures.

History and examples

As in other parts of mathematics, concrete problems and examples have played important roles in the development of algebra. Through the end of the nineteenth century many, perhaps most of these problems were in some way related to the theory of algebraic equations. Among major themes we can mention:

- solving of systems of linear equations, which led to matrices, determinants and linear algebra.
- attempts to find formulae for solutions of general polynomial equations of higher degree that resulted in discovery of groups as abstract manifestations of symmetry;
- and arithmetical investigations of quadratic and higher degree forms and diophantine equations, notably, in proving Fermat's last theorem, that directly produced the notions of a ring and ideal.

Numerous textbooks in abstract algebra start with axiomatic definitions of various algebraic structures and then proceed to establish their properties, creating a false impression that somehow in algebra axioms had come first and then served as a motivation and as a basis of further study. The true order of historical development was almost exactly the opposite. For example, the hypercomplex numbers of the nineteenth century had kinematic and physical motivations but challenged comprehension. Most theories that we now recognize as parts of algebra started as collections of disparate facts from various branches of mathematics, acquired a common theme that served as a core around which various results were grouped, and finally became unified on a basis of a common set of concepts. An archetypical example of this progressive synthesis can be seen in the theory of groups.

Early group theory

There were several threads in the early development of group theory, in modern language loosely corresponding to *number theory*, *theory of equations*, and *geometry*, of which we concentrate on the first two.

Leonhard Euler considered algebraic operations on numbers modulo an integer, modular arithmetic, proving his generalization of Fermat's little theorem. These investigations were taken much further by Carl Friedrich Gauss, who considered the structure of multiplicative groups of residues mod n and established many properties of cyclic and more general abelian groups that arise in this way. In his investigations of composition of binary quadratic forms, Gauss explicitly stated the associative law for the composition of forms, but like Euler before him, he seems to have been more interested in concrete results than in general theory. In 1870, Leopold Kronecker gave a definition of an abelian group in the context of ideal class groups of a number field, a far-reaching generalization of Gauss's work. It appears that he did not tie it with previous work on groups, in particular, permutation groups. In 1882 considering the same question, Heinrich M. Weber realized the connection and gave a similar definition that involved the cancellation property but omitted the existence of the inverse element, which was sufficient in his context (finite groups).

Permutations were studied by Joseph Lagrange in his 1770 paper *Réflexions sur la résolution algébrique des équations* devoted to solutions of algebraic equations, in which he introduced Lagrange resolvents. Lagrange's goal was to understand why equations of third and fourth degree admit formulae for solutions, and he identified as key objects permutations of the roots. An important novel step taken by Lagrange in this paper was the abstract view of the roots, i.e. as symbols and not as numbers. However, he did not consider composition of permutations. Serendipitously, the first edition of Edward Waring's *Meditationes Algebraicae* appeared in the same year, with an expanded version published in 1782. Waring proved the main theorem on symmetric functions, and specially considered the relation between the roots of a quartic equation and its resolvent cubic. *Mémoire sur la résolution des équations* of Alexandre Vandermonde (1771) developed the theory of symmetric functions from a slightly different angle, but like Lagrange, with the goal of understanding solvability of algebraic equations.

> *Kronecker claimed in 1888 that the study of modern algebra began with this first paper of Vandermonde. Cauchy states quite clearly that Vandermonde had priority over Lagrange for this remarkable idea which eventually led to the study of group theory.* [1]

Paolo Ruffini was the first person to develop the theory of permutation groups, and like his predecessors, also in the context of solving algebraic equations. His goal was to establish impossibility of algebraic solution to a general algebraic equation of degree greater than four. En route to this goal he introduced the notion of the order of an element of a group, conjugacy, the cycle decomposition of elements of permutation groups and the notions of primitive and imprimitive and proved some important theorems relating these concepts, such as

> *if G is a subgroup of S_5 whose order is divisible by 5 then G contains an element of order 5.*

Note, however, that he got by without formalizing the concept of a group, or even of a permutation group. The next step was taken by Évariste Galois in 1832, although his work remained unpublished until 1846, when he considered for the first time what we now call the *closure property* of a group of permutations, which he expressed as

> ... if in such a group one has the substitutions S and T then one has the substitution ST.

The theory of permutation groups received further far-reaching development in the hands of Augustin Cauchy and Camille Jordan, both through introduction of new concepts and, primarily, a great wealth of results about special classes of permutation groups and even some general theorems. Among other things, Jordan defined a notion of isomorphism, still in the context of permutation groups and, incidentally, it was he who put the term *group* in wide use.

The abstract notion of a group appeared for the first time in Arthur Cayley's papers in 1854. Cayley realized that a group need not be a permutation group (or even *finite*), and may instead consist of matrices, whose algebraic properties, such as multiplication and inverses, he systematically investigated in succeeding years. Much later

Cayley would revisit the question whether abstract groups were more general than permutation groups, and establish that, in fact, any group is isomorphic to a group of permutations.

Modern algebra

The end of 19th and the beginning of the 20th century saw a tremendous shift in methodology of mathematics. No longer satisfied with establishing properties of concrete objects, mathematicians started to turn their attention to general theory. For example, results about various groups of permutations came to be seen as instances of general theorems that concern a general notion of an *abstract group*. Questions of structure and classification of various mathematical objects came to forefront. These processes were occurring throughout all of mathematics, but became especially pronounced in algebra. Formal definition through primitive operations and axioms were proposed for many basic algebraic structures, such as groups, rings, and fields. The algebraic investigations of general fields by Ernst Steinitz and of commutative and then general rings by David Hilbert, Emil Artin and Emmy Noether, building up on the work of Ernst Kummer, Leopold Kronecker and Richard Dedekind, who had considered ideals in commutative rings, and of Georg Frobenius and Issai Schur, concerning representation theory of groups, came to define abstract algebra. These developments of the last quarter of the 19th century and the first quarter of 20th century were systematically exposed in Bartel van der Waerden's *Moderne algebra*, the two-volume monograph published in 1930–1931 that forever changed for the mathematical world the meaning of the word *algebra* from *the theory of equations* to the *theory of algebraic structures*.

An example

Abstract algebra facilitates the study of properties and patterns that seemingly disparate mathematical concepts have in common. For example, consider the distinct operations of function composition, $f(g(x))$, and of matrix multiplication, AB. These two operations have, in fact, the same structure. To see this, think about multiplying two square matrices, AB, by a one column vector, x. This defines a function equivalent to composing Ay with Bx: $Ay = A(Bx) = (AB)x$. Functions under composition and matrices under multiplication are examples of monoids. A set S and a binary operation over S, denoted by concatenation, form a monoid if the operation associates, $(ab)c = a(bc)$, and if there exists an $e \in S$, such that $ae = ea = a$.

Notes

[1] Vandermonde biography in Mac Tutor History of Mathematics Archive (http://www-groups.dcs.st-and.ac.uk/~history/Biographies/Vandermonde.html).

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- W. Keith Nicholson, *Introduction to abstract algebra*
- John R. Durbin, *Modern algebra : an introduction*

- Raymond A. Barnett, *Intermediate algebra; structure and use*

External links

- John Beachy: *Abstract Algebra On Line* (http://www.math.niu.edu/~beachy/aaol/contents.html), Comprehensive list of definitions and theorems.
- Edwin Connell " Elements of Abstract and Linear Algebra (http://www.math.miami.edu/~ec/book/)", Free online textbook.
- Fredrick M. Goodman: *Algebra: Abstract and Concrete* (http://www.math.uiowa.edu/~goodman/algebrabook.dir/algebrabook.html).
- Judson, Thomas W. (1997), *Abstract Algebra: Theory and Applications* (http://abstract.ups.edu) An introductory undergraduate text in the spirit of texts by Gallian or Herstein, covering groups, rings, integral domains, fields and Galois theory. Free downloadable PDF with open-source GFDL license.

Universal algebra

Universal algebra (sometimes called **general algebra**) is the field of mathematics that studies algebraic structures themselves, not examples ("models") of algebraic structures. For instance, rather than take particular groups as the object of study, in universal algebra one takes "the theory of groups" as an object of study.

Basic idea

From the point of view of universal algebra, an **algebra** (or **algebraic structure**) is a set A together with a collection of operations on A. An ***n*-ary operation** on A is a function that takes n elements of A and returns a single element of A. Thus, a 0-ary operation (or *nullary operation*) can be represented simply as an element of A, or a *constant*, often denoted by a letter like a. A 1-ary operation (or *unary operation*) is simply a function from A to A, often denoted by a symbol placed in front of its argument, like $\sim x$. A 2-ary operation (or *binary operation*) is often denoted by a symbol placed between its arguments, like $x * y$. Operations of higher or unspecified arity are usually denoted by function symbols, with the arguments placed in parentheses and separated by commas, like $f(x,y,z)$ or $f(x_1,...,x_n)$. Some researchers allow infinitary operations, such as $\bigwedge_{\alpha\in J} x_\alpha$ where J is an infinite index set, thus leading into the algebraic theory of complete lattices. One way of talking about an algebra, then, is by referring to it as an algebra of a certain type Ω, where Ω is an ordered sequence of natural numbers representing the arity of the operations of the algebra.

Equations

After the operations have been specified, the nature of the algebra can be further limited by axioms, which in universal algebra often take the form of identities, or **equational laws.** An example is the associative axiom for a binary operation, which is given by the equation $x * (y * z) = (x * y) * z$. The axiom is intended to hold for all elements x, y, and z of the set A.

Varieties

An algebraic structure which can be defined by identities is called a **variety,** and these are sufficiently important that some authors consider varieties the only object of study in universal algebra, while others consider them an object.

Restricting one's study to varieties rules out:

- Predicate logic, notably quantification, including existential quantification ($\exists$) and universal quantification ($\forall$)

- Relations, including inequalities, both $a \neq b$ and order relations

In this narrower definition, universal algebra can be seen as a special branch of model theory, in which we are typically dealing with structures having operations only (i.e. the type can have symbols for functions but not for relations other than equality), and in which the language used to talk about these structures uses equations only.

Not all algebraic structures in a wider sense fall into this scope. For example ordered groups are not studied in mainstream universal algebra because they involve an ordering relation.

A more fundamental restriction is that universal algebra cannot study the class of fields, because there is no type in which all field laws can be written as equations (inverses of elements are defined for all *non-zero* elements in a field, so inversion cannot simply be added to the type).

One advantage of this restriction is that the structures studied in universal algebra can be defined in any category which has *finite products*. For example, a topological group is just a group in the category of topological spaces.

Examples

Most of the usual algebraic systems of mathematics are examples of varieties, but not always in an obvious way – the usual definitions often involve quantification or inequalities.

Groups

To see how this works, let's consider the definition of a group. Normally a group is defined in terms of a single binary operation *, subject to these axioms:

- Associativity (as in the previous section): $x * (y * z) = (x * y) * z$.
- Identity element: There exists an element e such that for each element x, $e * x = x = x * e$.
- Inverse element: It can easily be seen that the identity element is unique. If we denote this unique identity element by e then for each x, there exists an element i such that $x * i = e = i * x$.

(Sometimes you will also see an axiom called "closure", stating that $x * y$ belongs to the set A whenever x and y do. But from a universal algebraist's point of view, that is already implied when you call * a binary operation.)

Now, this definition of a group is problematic from the point of view of universal algebra. The reason is that the axioms of the identity element and inversion are not stated purely in terms of equational laws but also have clauses involving the phrase "there exists ... such that ...". This is inconvenient; the list of group properties can be simplified to universally quantified equations if we add a nullary operation e and a unary operation ~ in addition to the binary operation *, then list the axioms for these three operations as follows:

- Associativity: $x * (y * z) = (x * y) * z$.
- Identity element: $e * x = x = x * e$.
- Inverse element: $x * (\sim x) = e = (\sim x) * x$.

(Of course, we usually write "x^{-1}" instead of "$\sim x$", which shows that the notation for operations of low arity is not *always* as given in the second paragraph.)

What has changed is that in the usual definition there are:

- a single binary operation (signature (2))
- 1 equational law (associativity)
- 2 quantified laws (identity and inverse)

...while in the universal algebra definition there are

- 3 operations: one binary, one unary, and one nullary (signature (2,1,0))
- 3 equational laws (associativity, identity, and inverse)
- no quantified laws

It's important to check that this really does capture the definition of a group. The reason that it might not is that specifying one of these universal groups might give more information than specifying one of the usual kind of group. After all, nothing in the usual definition said that the identity element e was *unique*; if there is another identity element e', then it's ambiguous which one should be the value of the nullary operator e. However, this is not a problem because identity elements can be proved to be always unique. The same thing is true of inverse elements. So the universal algebraist's definition of a group really is equivalent to the usual definition.

Basic constructions

We assume that the type, Ω, has been fixed. Then there are three basic constructions in universal algebra: homomorphic image, subalgebra, and product.

A homomorphism between two algebras A and B is a function $h: A \to B$ from the set A to the set B such that, for every operation f (of arity, say, n), $h(f_A(x_1,\ldots,x_n)) = f_B(h(x_1),\ldots,h(x_n))$. (Here, subscripts are placed on f to indicate whether it is the version of f in A or B. In theory, you could tell this from the context, so these subscripts are usually left off.) For example, if e is a constant (nullary operation), then $h(e_A) = e_B$. If ~ is a unary operation, then $h(\sim x) = \sim h(x)$. If * is a binary operation, then $h(x * y) = h(x) * h(y)$. And so on. A few of the things that can be done with homomorphisms, as well as definitions of certain special kinds of homomorphisms, are listed under the entry Homomorphism. In particular, we can take the homomorphic image of an algebra, $h(A)$.

A subalgebra of A is a subset of A that is closed under all the operations of A. A product of some set of algebraic structures is the cartesian product of the sets with the operations defined coordinatewise.

Some basic theorems

- The Isomorphism theorems, which encompass the isomorphism theorems of groups, rings, modules, etc.
- Birkhoff's HSP Theorem, which states that a class of algebras is a variety if and only if it is closed under homomorphic images, subalgebras, and arbitrary direct products.

Motivations and applications

In addition to its unifying approach, universal algebra also gives deep theorems and important examples and counterexamples. It provides a useful framework for those who intend to start the study of new classes of algebras. It can enable the use of methods invented for some particular classes of algebras to other classes of algebras, by recasting the methods in terms of universal algebra (if possible), and then interpreting these as applied to other classes. It has also provided conceptual clarification; as J.D.H. Smith puts it, *"What looks messy and complicated in a particular framework may turn out to be simple and obvious in the proper general one."*

In particular, universal algebra can be applied to the study of monoids, rings, and lattices. Before universal algebra came along, many theorems (most notably the isomorphism theorems) were proved separately in all of these fields, but with universal algebra, they can be proven once and for all for every kind of algebraic system.

Category theory and operads

A more generalised programme along these lines is carried out by category theory. Given a list of operations and axioms in universal algebra, the corresponding algebras and homomorphisms are the objects and morphisms of a category. Category theory applies to many situations where universal algebra does not, extending the reach of the theorems. Conversely, many theorems that hold in universal algebra do not generalise all the way to category theory. Thus both fields of study are useful.

A more recent development in category theory that generalizes operations is operad theory – an operad is a set of operations, similar to a universal algebra.

History

In Alfred North Whitehead's book *A Treatise on Universal Algebra,* published in 1898, the term *universal algebra* had essentially the same meaning that it has today. Whitehead credits William Rowan Hamilton and Augustus De Morgan as originators of the subject matter, and James Joseph Sylvester with coining the term itself[1] .

At the time structures such as Lie algebras and hyperbolic quaternions drew attention to the need to expand algebraic structures beyond the associatively multiplicative class. In a review Alexander Macfarlane wrote: "The main idea of the work is not unification of the several methods, nor generalization of ordinary algebra so as to include them, but rather the comparative study of their several structures." At the time George Boole's algebra of logic made a strong counterpoint to ordinary number algebra, so the term "universal" served to calm strained sensibilities.

Whitehead's early work sought to unify quaternions (due to Hamilton), Grassmann's Ausdehnungslehre, and Boole's algebra of logic. Whitehead wrote in his book:

> *"Such algebras have an intrinsic value for separate detailed study; also they are worthy of comparative study, for the sake of the light thereby thrown on the general theory of symbolic reasoning, and on algebraic symbolism in particular. The comparative study necessarily presupposes some previous separate study, comparison being impossible without knowledge."*[2]

Whitehead, however, had no results of a general nature. Work on the subject was minimal until the early 1930s, when Garrett Birkhoff and Øystein Ore began publishing on universal algebras. Developments in metamathematics and category theory in the 1940s and 1950s furthered the field, particularly the work of Abraham Robinson, Alfred Tarski, Andrzej Mostowski, and their students (Brainerd 1967).

In the period between 1935 and 1950, most papers were written along the lines suggested by Birkhoff's papers, dealing with free algebras, congruence and subalgebra lattices, and homomorphism theorems. Although the development of mathematical logic had made applications to algebra possible, they came about slowly; results published by Anatoly Maltsev in the 1940s went unnoticed because of the war. Tarski's lecture at the 1950 International Congress of Mathematicians in Cambridge ushered in a new period in which model-theoretic aspects were developed, mainly by Tarski himself, as well as C.C. Chang, Leon Henkin, Bjarni Jónsson, R. C. Lyndon, and others.

In the late 1950s, E. Marczewski[3] emphasized the importance of free algebras, leading to the publication of more than 50 papers on the algebraic theory of free algebras by Marczewski himself, together with J. Mycielski, W. Narkiewicz, W. Nitka, J. Płonka, S. Świerczkowski, K. Urbanik, and others.

Footnotes

[1] Grätzer, George. **Universal Algebra,** Van Nostrand Co., Inc., 1968, p. *v*.

[2] Quoted in Grätzer, George. **Universal Algebra,** Van Nostrand Co., Inc., 1968.

[3] Marczewski, E. "A general scheme of the notions of independence in mathematics." Bull. Acad. Polon. Sci. Ser. Sci. Math. Astronom. Phys. **6** (1958), 731–736.

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External links

- *Algebra Universalis* (http://www.birkhauser.ch/AU)—a journal dedicated to Universal Algebra.

Heyting algebra

In mathematics, **Heyting algebras** are special partially ordered sets that constitute a generalization of Boolean algebras, named after Arend Heyting. Heyting algebras arise as models of intuitionistic logic, a logic in which the law of excluded middle does not in general hold. Complete Heyting algebras are a central object of study in pointless topology.

Formal definition

A Heyting algebra H is a bounded lattice such that for all a and b in H there is a greatest element x of H such that

$$a \wedge x \leq b.$$

This element is the **relative pseudo-complement** of a with respect to b, and is denoted $a \to b$. We write 1 and 0 for the largest and the smallest element of H, respectively.

In any Heyting algebra, one defines the **pseudo-complement** $\neg x$ of any element x by setting $\neg x = (x \to 0)$. By definition, $a \wedge \neg a = 0$, and $\neg a$ is the largest element having this property. However, it is not in general true that $a \vee \neg a = 1$, thus $\neg$ is only a pseudo-complement, not a true complement, as would be the case in a Boolean algebra.

A **complete Heyting algebra** is a Heyting algebra that is a complete lattice.

A **subalgebra** of a Heyting algebra H is a subset H_1 of H containing 0 and 1 and closed under the operations $\wedge$, $\vee$ and $\to$. It follows that it is also closed under $\neg$. A subalgebra is made into a Heyting algebra by the induced operations.

Alternative definitions

Lattice-theoretic definitions

An equivalent definition of Heyting algebras can be given by considering the mappings

$$f_a : H \to H \text{ defined by } f_a(x) = a \wedge x,$$

for some fixed a in H. A bounded lattice H is a Heyting algebra if and only if all mappings f_a are the lower adjoint of a monotone Galois connection. In this case the respective upper adjoints g_a are given by $g_a(x) = a \to x$, where $\to$ is defined as above.

Yet another definition is as a residuated lattice whose monoid operation is $\wedge$. The monoid unit must then be the top element 1. Commutativity of this monoid implies that the two residuals coincide as $a \to b$.

Bounded lattice with an implication operation

Given a bounded lattice A with largest and smallest elements 1 and 0, and a binary operation $\to$, these together form a Heyting algebra if and only if the following hold:

1. $a \to a = 1$
2. $a \wedge (a \to b) = a \wedge b$
3. $b \wedge (a \to b) = b$
4. $a \to (b \wedge c) = (a \to b) \wedge (a \to c)$

where 4 is the distributive law for $\to$.

Characterization using the axioms of intuitionistic logic

This characterization of Heyting algebras makes the proof of the basic facts concerning the relationship between intuitionist propositional calculus and Heyting algebras immediate. (For these facts, see the sections "Provable identities" and "Universal constructions.") One should think of the element 1 as meaning, intuitively, "provably true." Compare with the axioms at Intuitionistic logic#Axiomatization.

Given a set A with three binary operations $\to$, $\wedge$ and $\vee$, and two distinguished elements 0 and 1, then A is a Heyting algebra for these operations (and the relation $\le$ defined by the condition that $a \le b$ when $a \to b = 1$) if and only if the following conditions hold for any elements x, y and z of A:

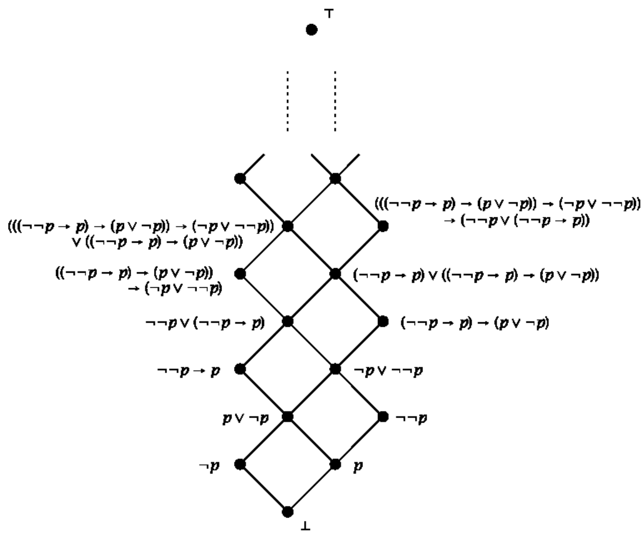
1. If $x \to y = 1$ and $y \to x = 1$ then $x = y$,
2. If $1 \to y = 1$, then $y = 1$,
3. $x \to (y \to x) = 1$,
4. $(x \to (y \to z)) \to ((x \to y) \to (x \to z)) = 1$,
5. $x \wedge y \to x = 1$,
6. $x \wedge y \to y = 1$,
7. $x \to (y \to (x \wedge y)) = 1$,
8. $x \to x \vee y = 1$,
9. $y \to x \vee y = 1$,
10. $(x \to z) \to ((y \to z) \to (x \vee y \to z)) = 1$,
11. $0 \to x = 1$.

Finally, we define $\neg x$ to be $x \to 0$.

Condition 1 says that equivalent formulas should be identified. Condition 2 says that provably true formulas are closed under modus ponens. Conditions 3 and 4 are *then* conditions. Conditions 5, 6 and 7 are *and* conditions. Conditions 8, 9 and 10 are *or* conditions. Condition 11 is a *false* condition.

Of course, if a different set of axioms were chosen for logic, we could modify ours accordingly.

Examples



The free Heyting algebra over one generator (aka Rieger–Nishimura lattice)

- Every Boolean algebra is a Heyting algebra, with $p \to q$ given by $\neg p \vee q$.
- Every totally ordered set that is a bounded lattice is also a Heyting algebra, where $p \to q$ is equal to q when $p>q$, and 1 otherwise.
- The simplest Heyting algebra that is not already a Boolean algebra is the totally ordered set {0, ½, 1} with $\to$ defined as above. Notice that ½ ∨ ¬½ = ½ ∨ (½ → 0) = ½ ∨ 0 = ½ falsifies the law of excluded middle.
- Every topology provides a complete Heyting algebra in the form of its open set lattice. In this case, the element $A \to B$ is the interior of the union of A^c and B, where A^c denotes the complement of the open set A. Not all complete Heyting algebras are of this form. These issues are studied in pointless topology, where complete Heyting algebras are also called **frames** or **locales**.
- The Lindenbaum algebra of propositional intuitionistic logic is a Heyting algebra.
- The global elements of the subobject classifier Ω of an elementary topos form a Heyting algebra; it is the Heyting algebra of truth values of the intuitionistic higher-order logic induced by the topos.

Properties

General properties

The ordering ≤ on a Heyting algebra H can be recovered from the operation $\to$ as follows: for any elements a, b of H, $a \le b$ if and only if $a \to b = 1$.

In contrast to some many-valued logics, Heyting algebras share the following property with Boolean algebras: if negation has a fixed point (i.e. $\neg a = a$ for some a), then the Heyting algebra is the trivial one-element Heyting algebra.

Provable identities

Given a formula $F(A_1, A_2,\ldots, A_n)$ of propositional calculus (using, in addition to the variables, the connectives $\wedge$, $\vee$, $\neg$, $\to$, and the constants 0 and 1), it is a fact, proved early on in any study of Heyting algebras, that the following two conditions are equivalent:

1. The formula F is provably true in intuitionist propositional calculus.
2. The identity $F(a_1, a_2,\ldots, a_n) = 1$ is true for any Heyting algebra H and any elements $a_1, a_2,\ldots, a_n \in H$.

The implication $1 \to 2$ is extremely useful and is the principal practical method for proving identities in Heyting algebras. In practice, one frequently uses the deduction theorem in such proofs.

Since for any a and b in a Heyting algebra H we have $a \le b$ if and only if $a \to b = 1$, it follows from $1 \to 2$ that whenever a formula a formula $F \to G$ is provably true, we have $F(a_1, a_2,\ldots, a_n) \le G(a_1, a_2,\ldots, a_n)$ for any Heyting algebra H, and any elements $a_1, a_2,\ldots, a_n \in H$. (It follows from the deduction theorem that $F \to G$ is provable [from nothing] if and only if G is a provable from F, that is, if G is a provable consequence of F.) In particular, if F and G

are provably equivalent, then $F(a_1, a_2,\ldots, a_n) = G(a_1, a_2,\ldots, a_n)$, since ≤ is an order relation.

1 → 2 can be proved by examining the logical axioms of the system of proof and verifying that their value is 1 in any Heyting algebra, and then verifying that the application of the rules of inference to expressions with value 1 in a Heyting algebra results in expressions with value 1. For example, let us choose the system of proof having modus ponens as its sole rule of inference, and whose axioms are the Hilbert-style ones given at Intuitionistic logic#Axiomatization. Then the facts to be verified follow immediately from the axiom-like definition of Heyting algebras given above.

1 → 2 also provides a method for proving that certain propositional formulas, though tautologies in classical logic, *cannot* be proved in intuitionist propositional logic. In order to prove that some formula $F(A_1, A_2,\ldots, A_n)$ is not provable, it is enough to exhibit a Heyting algebra H and elements $a_1, a_2,..., a_n \in H$ such that $F(a_1, a_2,\ldots, a_n) \neq 1$.

If one wishes to avoid mention of logic, then in practice it becomes necessary to prove as a lemma a version of the deduction theorem valid for Heyting algebras: for any elements a, b and c of a Heyting algebra H, we have $(a \wedge b) \to c = a \to (b \to c)$.

For more on the implication 2 → 1, see the section "Universal constructions" below.

Distributivity

Heyting algebras are always distributive. Specifically, we always have the identities

1. $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$
2. $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$

The distributive law is sometimes stated as an axiom, but in fact it follows from the existence of relative pseudo-complements. The reason is that, being the lower adjoint of a Galois connection, $\wedge$ preserves all existing suprema. Distributivity in turn is just the preservation of binary suprema by ∧.

By a similar argument, the following infinite distributive law holds in any complete Heyting algebra:

$$x \wedge \bigvee Y = \bigvee \{x \wedge y : y \in Y\}$$

for any element x in H and any subset Y of H. Conversely, any complete lattice satisfying the above infinite distributive law is a complete Heyting algebra, with

$$a \to b = \bigvee \{c \mid a \wedge c \leq b\}$$

being its relative pseudo-complement operation.

Regular and complemented elements

An element x of a Heyting algebra H is called **regular** if either of the following equivalent conditions hold:

1. $x = \neg\neg x.$
2. $x = \neg y$ for some $y \in H.$

The equivalence of these conditions can be restated simply as the identity $\neg\neg\neg x = \neg x$, valid for all $x \in H$.

Elements x and y of a Heyting algebra H are called **complements** to each other if $x \wedge y = 0$ and $x \vee y = 1$. If it exists, any such y is unique and must in fact be equal to $\neg x$. We call an element x **complemented** if it admits a complement. It is true that *if* x is complemented, then so is $\neg x$, and then x and $\neg x$ are complements to each other. However, confusingly, even if x is not complemented, $\neg x$ may nonetheless have a complement (not equal to x). In any Heyting algebra, the elements 0 and 1 are complements to each other. For instance, it is possible that $\neg x$ is 0 for every x different from 0, and 1 if $x = 0$, in which case 0 and 1 are the only regular elements.

Any complemented element of a Heyting algebra is regular, though the converse is not true in general. In particular, 0 and 1 are always regular.

For any Heyting algebra H, the following conditions are equivalent:

1. H is a Boolean algebra;
2. every x in H is regular;[1]
3. every x in H is complemented.[2]

In this case, the element $a \to b$ is equal to $\neg a \vee b$.

The regular (resp. complemented) elements of any Heyting algebra H constitute a Boolean algebra H_{reg} (resp. H_{comp}), in which the operations $\wedge$, $\neg$ and $\to$, as well as the constants 0 and 1, coincide with those of H. In the case of H_{comp}, the operation $\vee$ is also the same, hence H_{comp} is a subalgebra of H. In general however, H_{reg} will not be a subalgebra of H, because its join operation $\vee_{\text{reg}}$ may be differ from $\vee$. For $x, y \in H_{\text{reg}}$, we have $x \vee_{\text{reg}} y = \neg(\neg x \wedge \neg y)$. See below for necessary and sufficient conditions in order for $\vee_{\text{reg}}$ to coincide with $\vee$.

The De Morgan laws in a Heyting algebra

One of the two De Morgan laws is satisfied in every Heyting algebra, namely

$$\neg(x \vee y) = \neg x \wedge \neg y, \text{ for all } x, y \in H.$$

However, the other De Morgan law does not always hold. We have instead a weak de Morgan law:

$$\neg(x \wedge y) = \neg\neg(\neg x \vee \neg y), \text{ for all } x, y \in H.$$

The following statements are equivalent for all Heyting algebras H:

1. H satisfies both De Morgan laws,
2. $\neg(x \wedge y) = \neg x \vee \neg y$ for all $x, y \in H$,
3. $\neg(x \wedge y) = \neg x \vee \neg y$ for all regular $x, y \in H$,
4. $\neg\neg(x \vee y) = \neg\neg x \vee \neg\neg y$ for all $x, y \in H$,
5. $\neg\neg(x \vee y) = x \vee y$ for all regular $x, y \in H$,
6. $\neg(\neg x \wedge \neg y) = x \vee y$ for all regular $x, y \in H$,
7. $\neg x \vee \neg\neg x = 1$ for all $x \in H$.

Condition 2 is the other De Morgan law. Condition 6 says that the join operation $\vee_{\text{reg}}$ on the Boolean algebra H_{reg} of regular elements of H coincides with the operation $\vee$ of H. Condition 7 states that every regular element is complemented, i.e., $H_{\text{reg}} = H_{\text{comp}}$.

We prove the equivalence. Clearly $1 \to 2$, $2 \to 3$ and $4 \to 5$ are trivial. Furthermore, $3 \leftrightarrow 4$ and $5 \leftrightarrow 6$ result simply from the first De Morgan law and the definition of regular elements. We show that $6 \to 7$ by taking $\neg x$ and $\neg\neg x$ in place of x and y in 6 and using the identity $a \wedge \neg a = 0$. Notice that $2 \to 1$ follows from the first De Morgan law, and $7 \to 6$ results from the fact that the join operation $\vee$ on the subalgebra H_{comp} is just the restriction of $\vee$ to H_{comp}, taking into account the characterizations we have given of conditions 6 and 7. The implication $5 \to 2$ is a trivial consequence of the weak De Morgan law, taking $\neg x$ and $\neg y$ in place of x and y in 5.

Heyting algebras satisfying the above properties are related to De Morgan logic in the same way Heyting algebras in general are related to intuitionist logic.

Heyting algebra morphisms

Definition

Given two Heyting algebras H_1 and H_2 and a mapping $f : H_1 \to H_2$, we say that f is a **morphism** of Heyting algebras if, for any elements x and y in H_1, we have:

1. $f(0) = 0,$
2. $f(1) = 1,$
3. $f(x \wedge y) = f(x) \wedge f(y),$
4. $f(x \vee y) = f(x) \vee f(y),$
5. $f(x \to y) = f(x) \to f(y),$
6. $f(\neg x) = \neg f(x).$

We put condition 6 in brackets because it follows from the others, as $\neg x$ is just $x \to 0$, and one may or may not wish to consider ¬ to be a basic operation.

It follows from conditions 3 and 5 (or 1 alone, or 2 alone) that f is an increasing function, that is, that $f(x) \leq f(y)$ whenever $x \leq y$.

Assume H_1 and H_2 are structures with operations →, ∧, ∨ (and possibly ¬) and constants 0 and 1, and f is a surjective mapping from H_1 to H_2 with properties 1 through 5 (or 1 through 6) above. Then if H_1 is a Heyting algebra, so too is H_2. This follows from the characterization of Heyting algebras as bounded lattices (thought of as algebraic structures rather than partially ordered sets) with an operation → satisfying certain identities.

Properties

The identity map $f(x) = x$ from any Heyting algebra to itself is a morphism, and the composite $g \circ f$ of any two morphisms f and g is a morphism. Hence Heyting algebras form a category.

Examples

Given a Heyting algebra H and any subalgebra H_1, the inclusion mapping $i : H_1 \to H$ is a morphism.

For any Heyting algebra H, the map $x \mapsto \neg\neg x$ defines a morphism from H onto the Boolean algebra of its regular elements H_{reg}. This is *not* in general a morphism from H to itself, since the join operation of H_{reg} may be different from that of H.

Quotients

Let H be a Heyting algebra, and let $F \subseteq H$. We call F a **filter** on H if it satisfies the following properties:

1. $1 \in F,$
2. If $x, y \in F$ then $x \wedge y \in F,$
3. If $x \in F,\ y \in H,$ and $x \leq y$ then $y \in F.$

The intersection of any set of filters on H is again a filter. Therefore, given any subset S of H there is a smallest filter containing S. We call it the filter **generated** by S. If S is empty, $F = \{1\}$. Otherwise, F is equal to the set of x in H such that there exist $y_1, y_2, \ldots, y_n \in S$ with $y_1 \wedge y_2 \wedge \ldots \wedge y_n \leq x$.

If H is a Heyting algebra and F is a filter on H, we define a relation ~ on H as follows: we write $x \sim y$ whenever $x \to y$ and $y \to x$ both belong to F. Then ~ is an equivalence relation; we write H/F for the quotient set. There is a unique Heyting algebra structure on H/F such that the canonical surjection $p_F : H \to H/F$ becomes a Heyting algebra morphism. We call the Heyting algebra H/F the **quotient** of H by F.

Let S be a subset of a Heyting algebra H and let F be the filter generated by S. Then H/F satisfies the following universal property:

- Given any morphism of Heyting algebras $f : H \to H'$ satisfying $f(y) = 1$ for every $y \in S$, f factors uniquely through the canonical surjection $p_F : H \to H/F$. That is, there is a unique morphism $f' : H/F \to H'$ satisfying $f'p_F = f$. The morphism f' is said to be *induced* by f.

Let $f : H_1 \to H_2$ be a morphism of Heyting algebras. The **kernel** of f, written ker f, is the set $f^{-1}[\{1\}]$. It is a filter on H_1. (Care should be taken because this definition, if applied to a morphism of Boolean algebras, is dual to what would be called the kernel of the morphism viewed as a morphism of rings.) By the foregoing, f induces a morphism $f' : H_1/(\ker f) \to H_2$. It is an isomorphism of $H_1/(\ker f)$ onto the subalgebra $f[H_1]$ of H_2.

Universal constructions

Heyting algebra of propositional formulas in *n* variables up to intuitionist equivalence

The implication 2→1 in the section "Provable identities" is proved by showing that the result of the following construction is itself a Heyting algebra:

1. Consider the set L of propositional formulas in the variables $A_1, A_2, \ldots, A_n$.
2. Endow L with a preorder ≤ by defining $F \le G$ if G is an (intuitionist) logical consequence of F, that is, if G is provable from F. It is immediate that ≤ is a preorder.
3. Consider the equivalence relation $F \sim G$ induced by the preorder F≤G. (It is defined by $F \sim G$ if and only if $F \le G$ and $G \le F$. In fact, ~ is the relation of (intuitionist) logical equivalence.)
4. Let H_0 be the quotient set $L/\sim$. This will be the desired Heyting algebra.
5. We write $[F]$ for the equivalence class of a formula F. Operations →, ∧, ∨ and ¬ are defined in an obvious way on L. Verify that given formulas F and G, the equivalence classes $[F \to G]$, $[F \wedge G]$, $[F \vee G]$ and $[\neg F]$ depend only on $[F]$ and $[G]$. This defines operations →, ∧, ∨ and ¬ on the quotient set $H_0 = L/\sim$. Further define 1 to be the class of provably true statements, and set $0=[\bot]$.
6. Verify that H_0, together with these operations, is a Heyting algebra. We do this using the axiom-like definition of Heyting algebras. H_0 satisfies conditions THEN-1 through FALSE because all formulas of the given forms are axioms of intuitionist logic. MODUS-PONENS follows from the fact that if a formula $\top \to F$ is provably true, where ⊤ is provably true, then F is provably true (by application of the rule of inference modus ponens). Finally, EQUIV results from the fact that if $F \to G$ and $G \to F$ are both provably true, then F and G are provable from each other (by application of the rule of inference modus ponens), hence $[F]=[G]$.

As always under the axiom-like definition of Heyting algebras, we define ≤ on H_0 by the condition that $x \le y$ if and only if $x \to y = 1$. Since, by the deduction theorem, a formula $F \to G$ is provably true if and only if G is provable from F, it follows that $[F] \le [G]$ if and only if F≤G. In other words, ≤ is the order relation on $L/\sim$ induced by the preorder ≤ on L.

Free Heyting algebra on an arbitrary set of generators

In fact, the preceding construction can be carried out for any set of variables $\{A_i : i \in I\}$ (possibly infinite). One obtains in this way the *free* Heyting algebra on the variables $\{A_i\}$, which we will again denote by H_0. It is free in the sense that given any Heyting algebra H given together with a family of its elements $\langle a_i : i \in I \rangle$, there is a unique morphism $f : H_0 \to H$ satisfying $f([A_i]) = a_i$. The uniqueness of f is not difficult to see, and its existence results essentially from the implication 1→2 of the section "Provable identities" above, in the form of its corollary that whenever F and G are provably equivalent formulas, $F(\langle a_i \rangle) = G(\langle a_i \rangle)$ for any family of elements $\langle a_i \rangle$ in H.

Heyting algebra of formulas equivalent with respect to a theory *T*

Given a set of formulas T in the variables $\{A_i\}$, viewed as axioms, the same construction could have been carried out with respect to a relation $F \le G$ defined on L to mean that G is a provable consequence of F and the set of axioms T. Let us denote by H_T the Heyting algebra so obtained. Then H_T satisfies the same universal property as H_0 above, but with respect to Heyting algebras H and families of elements $\langle a_i \rangle$ satisfying the property that $J(\langle a_i \rangle)=1$ for any axiom $J(\langle A_i \rangle)$ in T. (Let us note that H_T, taken with the family of its elements $\langle [A_i] \rangle$, itself satisfies this property.) The existence and uniqueness of the morphism is proved the same way as for H_0, except that one must modify the implication 1→2 in "Provable identities" so that 1 reads "provably true *from T*," and 2 reads "any elements $a_1, a_2, \ldots, a_n$ in *H satisfying the formulas of T*."

The Heyting algebra H_T that we have just defined can be viewed as a quotient of the free Heyting algebra H_0 on the same set of variables, by applying the universal property of H_0 with respect to H_T, and the family of its elements $\langle [A_i] \rangle$.

Every Heyting algebra is isomorphic to one of the form H_T. To see this, let H be any Heyting algebra, and let $\langle a_i: i \in \mathrm{I} \rangle$ be a family of elements generating H (for example, any surjective family). Now consider the set T of formulas $J(\langle A_i \rangle)$ in the variables $\langle A_i: i \in \mathrm{I} \rangle$ such that $J(\langle a_i \rangle)=1$. Then we obtain a morphism $f: H_T \to H$ by the universal property of H_T, which is clearly surjective. It is not difficult to show that f is injective.

Comparison to Lindenbaum algebras

The constructions we have just given play an entirely analogous role with respect to Heyting algebras to that of Lindenbaum algebras with respect to Boolean algebras. In fact, The Lindenbaum algebra B_T in the variables $\{A_i\}$ with respect to the axioms T is just our $H_{T \cup T_1}$, where T_1 is the set of all formulas of the form $\neg\neg F \to F$, since the additional axioms of T_1 are the only ones that need to be added in order to make all classical tautologies provable.

Heyting algebras as applied to intuitionistic logic

If one interprets the axioms of the intuitionistic propositional logic as terms of a Heyting algebra, then they will evaluate to the largest element, 1, in *any* Heyting algebra under any assignment of values to the formula's variables. For instance, $(P \wedge Q) \to P$ is, by definition of the pseudo-complement, the largest element x such that $P \wedge Q \wedge x \le P$. This inequation is satisfied for any x, so the largest such x is 1.

Furthermore the rule of modus ponens allows us to derive the formula Q from the formulas P and P → Q. But in any Heyting algebra, if P has the value 1, and P → Q has the value 1, then it means that $P \wedge 1 \le Q$, and so $1 \wedge 1 \le Q$; it can only be that Q has the value 1.

This means that if a formula is deducible from the laws of intuitionistic logic, being derived from its axioms by way of the rule of modus ponens, then it will always have the value 1 in all Heyting algebras under any assignment of values to the formula's variables. However one can construct a Heyting algebra in which the value of Peirce's law is not always 1. Consider the 3-element algebra {0,½,1} as given above. If we assign ½ to P and 0 to Q, then the value of Peirce's law ((P → Q) → P) → P is ½. It follows that Peirce's law cannot be intuitionistically derived. See Curry–Howard isomorphism for the general context of what this implies in type theory.

The converse can be proven as well: if a formula always has the value 1, then it is deducible from the laws of intuitionistic logic, so the *intuitionistically valid* formulas are exactly those that always have a value of 1. This is similar to the notion that *classically valid* formulas are those formulas that have a value of 1 in the two-element Boolean algebra under any possible assignment of true and false to the formula's variables — that is, they are formulas which are tautologies in the usual truth-table sense. A Heyting algebra, from the logical standpoint, is then a generalization of the usual system of truth values, and its largest element 1 is analogous to 'true'. The usual two-valued logic system is a special case of a Heyting algebra, and the smallest non-trivial one, in which the only elements of the algebra are 1 (true) and 0 (false).

Word problem

The word problem on free Heyting algebras is difficult.[3] The only known results are that the free Heyting algebra on one generator is infinite, and that the free complete Heyting algebra on one generator exists (and has one more element than the free Heyting algebra).

Notes

[1] Rutherford (1965), Th.26.2 p.78.
[2] Rutherford (1965), Th.26.1 p.78.
[3] Peter T. Johnstone, *Stone Spaces*, (1982) Cambridge University Press, Cambridge, ISBN 0-521-23893-5. *(See paragraph 4.11)*

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External links

- Heyting algebra (GFDLed)

MV-algebra

In abstract algebra, a branch of pure mathematics, an **MV-algebra** is an algebraic structure with a binary operation $\oplus$, a unary operation $\neg$, and the constant 0, satisfying certain axioms. MV-algebras are models of Łukasiewicz logic; the letters MV refer to *multi-valued* logic of Łukasiewicz.

Definitions

An **MV-algebra** is an algebraic structure $\langle A, \oplus, \neg, 0\rangle$, consisting of

- a non-empty set A,
- a binary operation $\oplus$ on A,
- a unary operation $\neg$ on A, and
- a constant 0 denoting a fixed element of A,

which satisfies the following identities:

- $(x \oplus y) \oplus z = x \oplus (y \oplus z)$,
- $x \oplus 0 = x$,
- $x \oplus y = y \oplus x$,
- $\neg\neg x = x$,
- $x \oplus \neg 0 = \neg 0$, and
- $\neg(\neg x \oplus y) \oplus y = \neg(\neg y \oplus x) \oplus x$.

By virtue of the first three axioms, $\langle A, \oplus, 0\rangle$ is a commutative monoid. Being defined by identities, MV-algebras form a variety of algebras. The variety of MV-algebras is a subvariety of the variety of BL-algebras and contains all Boolean algebras.

An MV-algebra can equivalently be defined (Hájek 1998) as a prelinear commutative bounded integral residuated lattice $\langle L, \wedge, \vee, \otimes, \rightarrow, 0, 1\rangle$ satisfying the additional identity $x \vee y = (x \rightarrow y) \rightarrow y.$

Examples of MV-algebras

A simple numerical example is $A = [0, 1],$ with operations $x \oplus y = \min(x + y, 1)$ and $\neg x = 1 - x.$ In mathematical fuzzy logic, this MV-algebra is called the *standard MV-algebra*, as it forms the standard real-valued semantics of Łukasiewicz logic.

The *trivial* MV-algebra has the only element 0 and the operations defined in the only possible way, $0 \oplus 0 = 0$ and $\neg 0 = 0.$

The *two-element* MV-algebra is actually the two-element Boolean algebra $\{0, 1\},$ with $\oplus$ coinciding with Boolean disjunction and $\neg$ with Boolean negation.

Other finite linearly ordered MV-algebras are obtained by restricting the universe and operations of the standard MV-algebra to the set of $n + 1$ equidistant real numbers between 0 and 1 (both included), that is, the set $\{0, 1/n, 2/n, \dots, 1\},$ which is closed under the operations $\oplus$ and $\neg$ of the standard MV-algebra.

Another important example is *Chang's MV-algebra*, consisting just of infinitesimals (with the order type ω) and their co-infinitesimals.

Relation to Łukasiewicz logic

Chang devised MV-algebras to study multi-valued logics, introduced by Jan Łukasiewicz in 1920. In particular, MV-algebras form the algebraic semantics of Łukasiewicz logic, as described below.

Given an MV-algebra *A*, an *A*-valuation is a homomorphism from the algebra of propositional formulas (in the language consisting of $\oplus, \neg,$ and 0) into *A*. Formulas mapped to 1 (or $\neg$ 0) for all *A*-valuations are called *A*-tautologies. If the standard MV-algebra over [0,1] is employed, the set of all [0,1]-tautologies determines so-called infinite-valued Łukasiewicz logic.

Chang's (1958, 1959) completeness theorem states that any MV-algebra equation holding in the standard MV-algebra over the interval [0,1] will hold in every MV-algebra. Algebraically, this means that the standard MV-algebra generates the variety of all MV-algebras. Equivalently, Chang's completeness theorem says that MV-algebras characterize infinite-valued Łukasiewicz logic, defined as the set of [0,1]-tautologies.

The way the [0,1] MV-algebra characterizes all possible MV-algebras parallels the well-known fact that identities holding in the two-element Boolean algebra hold in all possible Boolean algebras. Moreover, MV-algebras characterize infinite-valued Łukasiewicz logic in a manner analogous to the way that Boolean algebras characterize classical bivalent logic (see Lindenbaum-Tarski algebra).

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External links

- Stanford Encyclopedia of Philosophy: "Many-valued logic [2]" -- by Siegfried Gottwald.

Group theory

In mathematics and abstract algebra, **group theory** studies the algebraic structures known as groups. The concept of a group is central to abstract algebra: other well-known algebraic structures, such as rings, fields, and vector spaces can all be seen as groups endowed with additional operations and axioms. Groups recur throughout mathematics, and the methods of group theory have strongly influenced many parts of algebra. Linear algebraic groups and Lie groups are two branches of group theory that have experienced tremendous advances and have become subject areas in their own right.

Various physical systems, such as crystals and the hydrogen atom, can be modelled by symmetry groups. Thus group theory and the closely related representation theory have many applications in physics and chemistry.

One of the most important mathematical achievements of the 20th century was the collaborative effort, taking up more than 10,000 journal pages and mostly published between 1960 and 1980, that culminated in a complete classification of finite simple groups.

History

Group theory has three main historical sources: number theory, the theory of algebraic equations, and geometry. The number-theoretic strand was begun by Leonhard Euler, and developed by Gauss's work on modular arithmetic and additive and multiplicative groups related to quadratic fields. Early results about permutation groups were obtained by Lagrange, Ruffini, and Abel in their quest for general solutions of polynomial equations of high degree. Évariste Galois coined the term “group” and established a connection, now known as Galois theory, between the nascent theory of groups and field theory. In geometry, groups first became important in projective geometry and, later, non-Euclidean geometry. Felix Klein's Erlangen program famously proclaimed group theory to be the organizing principle of geometry.

Galois, in the 1830s, was the first to employ groups to determine the solvability of polynomial equations. Arthur Cayley and Augustin Louis Cauchy pushed these investigations further by creating the theory of permutation group. The second historical source for groups stems from geometrical situations. In an attempt to come to grips with possible geometries (such as euclidean, hyperbolic or projective geometry) using group theory, Felix Klein initiated the Erlangen programme. Sophus Lie, in 1884, started using groups (now called Lie groups) attached to analytic problems. Thirdly, groups were (first implicitly and later explicitly) used in algebraic number theory.

The different scope of these early sources resulted in different notions of groups. The theory of groups was unified starting around 1880. Since then, the impact of group theory has been ever growing, giving rise to the birth of abstract algebra in the early 20th century, representation theory, and many more influential spin-off domains. The classification of finite simple groups is a vast body of work from the mid 20th century, classifying all the finite simple groups.

Main classes of groups

The range of groups being considered has gradually expanded from finite permutation groups and special examples of matrix groups to abstract groups that may be specified through a presentation by generators and relations.

Permutation groups

The first class of groups to undergo a systematic study was permutation groups. Given any set X and a collection G of bijections of X into itself (known as *permutations*) that is closed under compositions and inverses, G is a group acting on X. If X consists of n elements and G consists of *all* permutations, G is the symmetric group S_n; in general, G is a subgroup of the symmetric group of X. An early construction due to Cayley exhibited any group as a permutation group, acting on itself ($X = G$) by means of the left regular representation.

In many cases, the structure of a permutation group can be studied using the properties of its action on the corresponding set. For example, in this way one proves that for $n \geq 5$, the alternating group A_n is simple, i.e. does not admit any proper normal subgroups. This fact plays a key role in the impossibility of solving a general algebraic equation of degree $n \geq 5$ in radicals.

Matrix groups

The next important class of groups is given by *matrix groups*, or linear groups. Here G is a set consisting of invertible matrices of given order n over a field K that is closed under the products and inverses. Such a group acts on the n-dimensional vector space K^n by linear transformations. This action makes matrix groups conceptually similar to permutation groups, and geometry of the action may be usefully exploited to establish properties of the group G.

Transformation groups

Permutation groups and matrix groups are special cases of transformation groups: groups that act on a certain space X preserving its inherent structure. In the case of permutation groups, X is a set; for matrix groups, X is a vector space. The concept of a transformation group is closely related with the concept of a symmetry group: transformation groups frequently consist of *all* transformations that preserve a certain structure.

The theory of transformation groups forms a bridge connecting group theory with differential geometry. A long line of research, originating with Lie and Klein, considers group actions on manifolds by homeomorphisms or diffeomorphisms. The groups themselves may be discrete or continuous.

Abstract groups

Most groups considered in the first stage of the development of group theory were "concrete", having been realized through numbers, permutations, or matrices. It was not until the late nineteenth century that the idea of an abstract group as a set with operations satisfying a certain system of axioms began to take hold. A typical way of specifying an abstract group is through a presentation by *generators and relations*,

$$G = \langle S|R \rangle.$$

A significant source of abstract groups is given by the construction of a *factor group*, or quotient group, G/H, of a group G by a normal subgroup H. Class groups of algebraic number fields were among the earliest examples of factor groups, of much interest in number theory. If a group G is a permutation group on a set X, the factor group G/H is no longer acting on X; but the idea of an abstract group permits one not to worry about this discrepancy.

The change of perspective from concrete to abstract groups makes it natural to consider properties of groups that are independent of a particular realization, or in modern language, invariant under isomorphism, as well as the classes of group with a given such property: finite groups, periodic groups, simple groups, solvable groups, and so on. Rather than exploring properties of an individual group, one seeks to establish results that apply to a whole class of groups.

The new paradigm was of paramount importance for the development of mathematics: it foreshadowed the creation of abstract algebra in the works of Hilbert, Emil Artin, Emmy Noether, and mathematicians of their school.

Topological and algebraic groups

An important elaboration of the concept of a group occurs if G is endowed with additional structure, notably, of a topological space, differentiable manifold, or algebraic variety. If the group operations m (multiplication) and i (inversion),

$$m : G \times G \to G, (g, h) \mapsto gh, \quad i : G \to G, g \mapsto g^{-1},$$

are compatible with this structure, i.e. are continuous, smooth or regular (in the sense of algebraic geometry) maps then G becomes a topological group, a Lie group, or an algebraic group.[1]

The presence of extra structure relates these types of groups with other mathematical disciplines and means that more tools are available in their study. Topological groups form a natural domain for abstract harmonic analysis, whereas Lie groups (frequently realized as transformation groups) are the mainstays of differential geometry and unitary representation theory. Certain classification questions that cannot be solved in general can be approached and resolved for special subclasses of groups. Thus, compact connected Lie groups have been completely classified. There is a fruitful relation between infinite abstract groups and topological groups: whenever a group Γ can be realized as a lattice in a topological group G, the geometry and analysis pertaining to G yield important results about Γ. A comparatively recent trend in the theory of finite groups exploits their connections with compact topological groups (profinite groups): for example, a single p-adic analytic group G has a family of quotients which are finite p-groups of various orders, and properties of G translate into the properties of its finite quotients.

Combinatorial and geometric group theory

Groups can be described in different ways. Finite groups can be described by writing down the group table consisting of all possible multiplications $g \bullet h$. A more important way of defining a group is by *generators and relations*, also called the *presentation* of a group. Given any set F of generators $\{g_i\}_{i \in I}$, the free group generated by F surjects onto the group G. The kernel of this map is called subgroup of relations, generated by some subset D. The presentation is usually denoted by $\langle F \mid D \rangle$. For example, the group $\mathbf{Z} = \langle a \mid \rangle$ can be generated by one element a (equal to +1 or −1) and no relations, because $n \cdot 1$ never equals 0 unless n is zero. A string consisting of generator symbols is called a *word*.

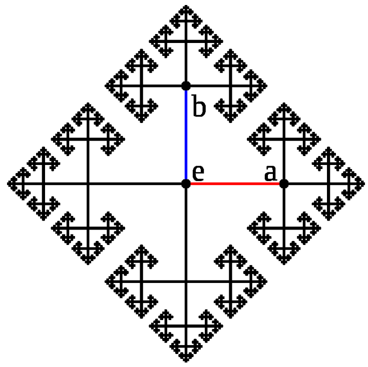
Combinatorial group theory studies groups from the perspective of generators and relations.[2] It is particularly useful where finiteness assumptions are satisfied, for example finitely generated groups, or finitely presented groups (i.e. in addition the relations are finite). The area makes use of the connection of graphs via their fundamental groups. For example, one can show that every subgroup of a free group is free.

There are several natural questions arising from giving a group by its presentation. The *word problem* asks whether two words are effectively the same group element. By relating the problem to Turing machines, one can show that there is in general no algorithm solving this task. An equally difficult problem is, whether two groups given by different presentations are actually isomorphic. For example $\mathbf{Z}$ can also be presented by

$$\langle x, y \mid xyxyx = 1 \rangle$$

and it is not obvious (but true) that this presentation is isomorphic to the standard one above.

Geometric group theory attacks these problems from a geometric viewpoint, either by viewing groups as geometric objects, or by finding suitable geometric objects a group acts on.[3] The first idea is made precise by means of the Cayley graph, whose vertices correspond to group elements and edges correspond to right multiplication in the group. Given two elements, one constructs the word metric given by the length of the minimal path between the elements. A theorem of Milnor and Svarc then says that given a group G acting in a reasonable manner on a metric space X, for example a compact manifold, then G is quasi-isometric (i.e. looks similar from the far) to the space X.



The Cayley graph of ⟨ x, y ∣ ⟩, the free group of rank 2.

Representation of groups

Saying that a group G *acts* on a set X means that every element defines a bijective map on a set in a way compatible with the group structure. When X has more structure, it is useful to restrict this notion further: a representation of G on a vector space V is a group homomorphism:

$$\varrho : G \to GL(V),$$

where $GL(V)$ consists of the invertible linear transformations of V. In other words, to every group element g is assigned an automorphism $\varrho(g)$ such that $\varrho(g) \circ \varrho(h) = \varrho(gh)$ for any h in G.

This definition can be understood in two directions, both of which give rise to whole new domains of mathematics.[4] On the one hand, it may yield new information about the group G: often, the group operation in G is abstractly given, but via ϱ, it corresponds to the multiplication of matrices, which is very explicit.[5] On the other hand, given a well-understood group acting on a complicated object, this simplifies the study of the object in question. For example, if G is finite, it is known that V above decomposes into irreducible parts. These parts in turn are much more easily manageable than the whole V (via Schur's lemma).

Given a group G, representation theory then asks what representations of G exist. There are several settings, and the employed methods and obtained results are rather different in every case: representation theory of finite groups and representations of Lie groups are two main subdomains of the theory. The totality of representations is governed by the group's characters. For example, Fourier polynomials can be interpreted as the characters of $U(1)$, the group of complex numbers of absolute value 1, acting on the L^2-space of periodic functions.

Connection of groups and symmetry

Given a structured object X of any sort, a symmetry is a mapping of the object onto itself which preserves the structure. This occurs in many cases, for example

1. If X is a set with no additional structure, a symmetry is a bijective map from the set to itself, giving rise to permutation groups.
2. If the object X is a set of points in the plane with its metric structure or any other metric space, a symmetry is a bijection of the set to itself which preserves the distance between each pair of points (an isometry). The corresponding group is called isometry group of X.
3. If instead angles are preserved, one speaks of conformal maps. Conformal maps give rise to Kleinian groups, for example.
4. Symmetries are not restricted to geometrical objects, but include algebraic objects as well. For instance, the equation

$$x^2 - 3 = 0$$

has the two solutions $+\sqrt{3}$, and $-\sqrt{3}$. In this case, the group that exchanges the two roots is the Galois group belonging to the equation. Every polynomial equation in one variable has a Galois group, that is a certain permutation group on its roots.

The axioms of a group formalize the essential aspects of symmetry. Symmetries form a group: they are closed because if you take a symmetry of an object, and then apply another symmetry, the result will still be a symmetry. The identity keeping the object fixed is always a symmetry of an object. Existence of inverses is guaranteed by undoing the symmetry and the associativity comes from the fact that symmetries are functions on a space, and composition of functions are associative.

Frucht's theorem says that every group is the symmetry group of some graph. So every abstract group is actually the symmetries of some explicit object.

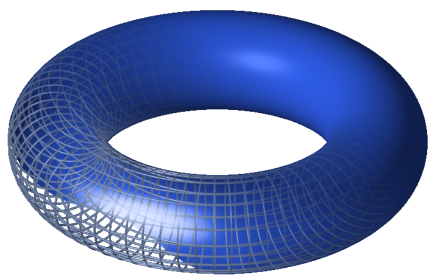
The saying of "preserving the structure" of an object can be made precise by working in a category. Maps preserving the structure are then the morphisms, and the symmetry group is the automorphism group of the object in question.

Applications of group theory

Applications of group theory abound. Almost all structures in abstract algebra are special cases of groups. Rings, for example, can be viewed as abelian groups (corresponding to addition) together with a second operation (corresponding to multiplication). Therefore group theoretic arguments underlie large parts of the theory of those entities.

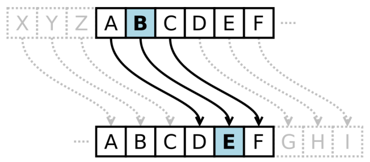
Galois theory uses groups to describe the symmetries of the roots of a polynomial (or more precisely the automorphisms of the algebras generated by these roots). The fundamental theorem of Galois theory provides a link between algebraic field extensions and group theory. It gives an effective criterion for the solvability of polynomial equations in terms of the solvability of the corresponding Galois group. For example, S_5, the symmetric group in 5 elements, is not solvable which implies that the general quintic equation cannot be solved by radicals in the way equations of lower degree can. The theory, being one of the historical roots of group theory, is still fruitfully applied to yield new results in areas such as class field theory.

Algebraic topology is another domain which prominently associates groups to the objects the theory is interested in. There, groups are used to describe certain invariants of topological spaces. They are called "invariants" because they are defined in such a way that they do not change if the space is subjected to some deformation. For example, the fundamental group "counts" how many paths in the space are essentially different. The Poincaré conjecture, proved in 2002/2003 by Grigori Perelman is a prominent application of this idea. The influence is not unidirectional, though. For example, algebraic topology makes use of Eilenberg–MacLane spaces which are spaces with prescribed homotopy groups. Similarly algebraic K-theory stakes in a crucial way on classifying spaces of groups. Finally, the name of the torsion subgroup of an infinite group shows the legacy of topology in group theory.



A torus. Its abelian group structure is induced from the map $\mathbf{C} \to \mathbf{C}/\mathbf{Z}+\tau\mathbf{Z}$, where τ is a parameter.

Algebraic geometry and cryptography likewise uses group theory in many ways. Abelian varieties have been introduced above. The presence of the group operation yields additional information which makes these varieties particularly accessible. They also often serve as a test for new conjectures.[6] The one-dimensional case, namely elliptic curves is studied in particular detail. They are both theoretically and practically intriguing.[7] Very large groups of prime order constructed in Elliptic-Curve Cryptography serve for public key cryptography. Cryptographical methods of this kind



The cyclic group $\mathbf{Z}_{26}$ underlies Caesar's cipher.

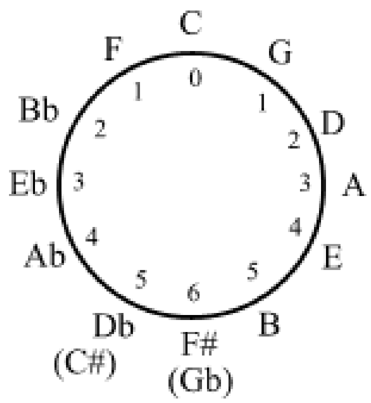
benefit from the flexibility of the geometric objects, hence their group structures, together with the complicated structure of these groups, which make the discrete logarithm very hard to calculate. One of the earliest encryption protocols, Caesar's cipher, may also be interpreted as a (very easy) group operation. In another direction, toric varieties are algebraic varieties acted on by a torus. Toroidal embeddings have recently led to advances in algebraic geometry, in particular resolution of singularities.[8]

Algebraic number theory is a special case of group theory, thereby following the rules of the latter. For example, Euler's product formula

$$\sum_{n\geq 1}\frac{1}{n^s} = \prod_{p \text{ prime}} \frac{1}{1-p^{-s}}$$

captures the fact that any integer decomposes in a unique way into primes. The failure of this statement for more general rings gives rise to class groups and regular primes, which feature in Kummer's treatment of Fermat's Last Theorem.

- The concept of the Lie group (named after mathematician Sophus Lie) is important in the study of differential equations and manifolds; they describe the symmetries of continuous geometric and analytical structures. Analysis on these and other groups is called harmonic analysis. Haar measures, that is integrals invariant under the translation in a Lie group, are used for pattern recognition and other image processing techniques.[9]
- In combinatorics, the notion of permutation group and the concept of group action are often used to simplify the counting of a set of objects; see in particular Burnside's lemma.
- The presence of the 12-periodicity in the circle of fifths yields applications of elementary group theory in musical set theory.
- In physics, groups are important because they describe the symmetries which the laws of physics seem to obey. Physicists are very interested in group representations, especially of Lie groups, since these representations often point the way to the "possible" physical theories. Examples of the use of groups in physics include the Standard Model, gauge theory, the Lorentz group, and the Poincaré group.
- In chemistry and materials science, groups are used to classify crystal structures, regular polyhedra, and the symmetries of molecules. The assigned point groups can then be used to determine physical properties (such as polarity and chirality), spectroscopic properties (particularly useful for Raman spectroscopy and infrared spectroscopy), and to construct molecular orbitals.



The circle of fifths may be endowed with a cyclic group structure

See also

- Group (mathematics)
- Glossary of group theory
- List of group theory topics

Notes

[1] This process of imposing extra structure has been formalized through the notion of a group object in a suitable category. Thus Lie groups are group objects in the category of differentiable manifolds and affine algebraic groups are group objects in the category of affine algebraic varieties.
[2] Schupp & Lyndon 2001
[3] La Harpe 2000
[4] Such as group cohomology or equivariant K-theory.
[5] In particular, if the representation is faithful.
[6] For example the Hodge conjecture (in certain cases).
[7] See the Birch-Swinnerton-Dyer conjecture, one of the millennium problems
[8] Abramovich, Dan; Karu, Kalle; Matsuki, Kenji; Wlodarczyk, Jaroslaw (2002), "Torification and factorization of birational maps", *Journal of the American Mathematical Society* **15** (3): 531–572, doi:10.1090/S0894-0347-02-00396-X, MR1896232
[9] Lenz, Reiner (1990), *Group theoretical methods in image processing* (http://webstaff.itn.liu.se/~reile/LNCS413/index.htm), Lecture Notes in Computer Science, **413**, Berlin, New York: Springer-Verlag, doi:10.1007/3-540-52290-5, ISBN 978-0-387-52290-6,

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External links

- History of the abstract group concept (http://www-history.mcs.st-andrews.ac.uk/history/HistTopics/Abstract_groups.html)
- Higher dimensional group theory (http://www.bangor.ac.uk/r.brown/hdaweb2.htm) This presents a view of group theory as level one of a theory which extends in all dimensions, and has applications in homotopy theory and to higher dimensional nonabelian methods for local-to-global problems.
- Plus teacher and student package: Group Theory (http://plus.maths.org/issue48/package/index.html) This package brings together all the articles on group theory from *Plus*, the online mathematics magazine produced by the Millennium Mathematics Project at the University of Cambridge, exploring applications and recent breakthroughs, and giving explicit definitions and examples of groups.
- US Naval Academy group theory guide (http://www.usna.edu/Users/math/wdj/tonybook/gpthry/node1.html) A general introduction to group theory with exercises written by Tony Gaglione.

Abelian group

Concepts in group theory
category of groups
subgroups, normal subgroups
group homomorphisms, kernel, image, quotient
direct product, direct sum
semidirect product, wreath product
Types of groups
simple, finite, infinite
discrete, continuous
multiplicative, additive
cyclic, abelian, dihedral
nilpotent, solvable
list of group theory topics
glossary of group theory

An **abelian group**, also called a **commutative group**, is a group in which the result of applying the group operation to two group elements does not depend on their order (the axiom of commutativity). Abelian groups generalize the arithmetic of addition of integers. They are named after Niels Henrik Abel.[1]

The concept of an abelian group is one of the first concepts encountered in undergraduate abstract algebra, with many other basic objects, such as a module and a vector space, being its refinements. The theory of abelian groups is generally simpler than that of their non-abelian counterparts, and finite abelian groups are very well understood. On the other hand, the theory of infinite abelian groups is an area of current research.

Definition

An abelian group is a set, A, together with an operation "•" that combines any two elements a and b to form another element denoted $a \bullet b$. The symbol "•" is a general placeholder for a concretely given operation. To qualify as an abelian group, the set and operation, $(A, \bullet)$, must satisfy five requirements known as the *abelian group axioms*:

Closure

For all a, b in A, the result of the operation $a \bullet b$ is also in A.

Associativity

For all a, b and c in A, the equation $(a \bullet b) \bullet c = a \bullet (b \bullet c)$ holds.

Identity element

There exists an element e in A, such that for all elements a in A, the equation $e \bullet a = a \bullet e = a$ holds.

Inverse element

For each a in A, there exists an element b in A such that $a \bullet b = b \bullet a = e$, where e is the identity element.

Commutativity

For all a, b in A, $a \bullet b = b \bullet a$.

More compactly, an abelian group is a commutative group. A group in which the group operation is not commutative is called a "non-abelian group" or "non-commutative group".

Facts

Notation

There are two main notational conventions for abelian groups — additive and multiplicative.

Convention	Operation	Identity	Powers	Inverse
Addition	$x + y$	0	nx	$-x$
Multiplication	$x * y$ or xy	e or 1	x^n	x^{-1}

Generally, the multiplicative notation is the usual notation for groups, while the additive notation is the usual notation for modules. The additive notation may also be used to emphasize that a particular group is abelian, whenever both abelian and non-abelian groups are considered.

Multiplication table

To verify that a finite group is abelian, a table (matrix) - known as a Cayley table - can be constructed in a similar fashion to a multiplication table. If the group is $G = \{g_1 = e, g_2, ..., g_n\}$ under the operation $\cdot$, the (i, j)'th entry of this table contains the product $g_i \cdot g_j$. The group is abelian if and only if this table is symmetric about the main diagonal (i.e. if the matrix is a symmetric matrix).

This is true since if the group is abelian, then $g_i \cdot g_j = g_j \cdot g_i$. This implies that the (i, j)'th entry of the table equals the (j, i)'th entry - i.e. the table is symmetric about the main diagonal.

Examples

- For the integers and the operation addition "+", denoted (**Z**,+), the operation + combines any two integers to form a third integer, addition is associative, zero is the additive identity, every integer n has an additive inverse, $-n$, and the addition operation is commutative since $m + n = n + m$ for any two integers m and n.
- Every cyclic group G is abelian, because if x, y are in G, then $xy = a^m a^n = a^{m+n} = a^{n+m} = a^n a^m = yx$. Thus the integers, **Z**, form an abelian group under addition, as do the integers modulo n, **Z**/n**Z**.
- Every ring is an abelian group with respect to its addition operation. In a commutative ring the invertible elements, or units, form an abelian multiplicative group. In particular, the real numbers are an abelian group under addition, and the nonzero real numbers are an abelian group under multiplication.
- Every subgroup of an abelian group is normal, so each subgroup gives rise to a quotient group. Subgroups, quotients, and direct sums of abelian groups are again abelian.

In general, matrices, even invertible matrices, do not form an abelian group under multiplication because matrix multiplication is generally not commutative. However, some groups of matrices are abelian groups under matrix multiplication - one example is the group of 2x2 rotation matrices.

Historical remarks

Abelian groups were named for Norwegian mathematician Niels Henrik Abel by Camille Jordan because Abel found that the commutativity of the group of an equation implies its roots are solvable by radicals. See Section 6.5 of Cox (2004) for more information on the historical background.

Properties

If n is a natural number and x is an element of an abelian group G written additively, then nx can be defined as $x + x + ... + x$ (n summands) and $(-n)x = -(nx)$. In this way, G becomes a module over the ring **Z** of integers. In fact, the modules over **Z** can be identified with the abelian groups.

Theorems about abelian groups (i.e. modules over the principal ideal domain **Z**) can often be generalized to theorems about modules over an arbitrary principal ideal domain. A typical example is the classification of finitely generated abelian groups which is a specialization of the structure theorem for finitely generated modules over a principal ideal domain. In the case of finitely generated abelian groups, this theorem guarantees that an abelian group splits as a direct sum of a torsion group and a free abelian group. The former may be written as a direct sum of finitely many groups of the form $\mathbf{Z}/p^k\mathbf{Z}$ for p prime, and the latter is a direct sum of finitely many copies of **Z**.

If $f, g : G \to H$ are two group homomorphisms between abelian groups, then their sum $f + g$, defined by $(f + g)(x) = f(x) + g(x)$, is again a homomorphism. (This is not true if H is a non-abelian group.) The set Hom(G, H) of all group homomorphisms from G to H thus turns into an abelian group in its own right.

Somewhat akin to the dimension of vector spaces, every abelian group has a *rank*. It is defined as the cardinality of the largest set of linearly independent elements of the group. The integers and the rational numbers have rank one, as well as every subgroup of the rationals.

Finite abelian groups

Cyclic groups of integers modulo n, $\mathbf{Z}/n\mathbf{Z}$, were among the first examples of groups. It turns out that an arbitrary finite abelian group is isomorphic to a direct sum of finite cyclic groups of prime power order, and these orders are uniquely determined, forming a complete system of invariants. The automorphism group of a finite abelian group can be described directly in terms of these invariants. The theory had been first developed in the 1879 paper of Georg Frobenius and Ludwig Stickelberger and later was both simplified and generalized to finitely generated modules over a principal ideal domain, forming an important chapter of linear algebra.

Classification

The **fundamental theorem of finite abelian groups** states that every finite abelian group G can be expressed as the direct sum of cyclic subgroups of prime-power order. This is a special case of the fundamental theorem of finitely generated abelian groups when G has zero rank.

The cyclic group $\mathbb{Z}_{mn}$ of order mn is isomorphic to the direct sum of $\mathbb{Z}_m$ and $\mathbb{Z}_n$ if and only if m and n are coprime. It follows that any finite abelian group G is isomorphic to a direct sum of the form

$$\mathbb{Z}_{k_1} \oplus \cdots \oplus \mathbb{Z}_{k_u}$$

in either of the following canonical ways:

- the numbers $k_1,...,k_u$ are powers of primes
- k_1 divides k_2, which divides k_3, and so on up to k_u.

For example, $\mathbb{Z}_{15}$ can be expressed as the direct sum of two cyclic subgroups of order 3 and 5: $\mathbb{Z}_{15} \cong \{0, 5, 10\} \oplus \{0, 3, 6, 9, 12\}$. The same can be said for any abelian group of order 15, leading to the remarkable conclusion that all abelian groups of order 15 are isomorphic.

For another example, every abelian group of order 8 is isomorphic to either $\mathbb{Z}_8$ (the integers 0 to 7 under addition modulo 8), $\mathbb{Z}_4 \oplus \mathbb{Z}_2$ (the odd integers 1 to 15 under multiplication modulo 16), or $\mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$.

See also list of small groups for finite abelian groups of order 16 or less.

Automorphisms

One can apply the fundamental theorem to count (and sometimes determine) the automorphisms of a given finite abelian group *G*. To do this, one uses the fact (which will not be proved here) that if *G* splits as a direct sum $H \oplus K$ of subgroups of coprime order, then $\mathrm{Aut}(H \oplus K) \cong \mathrm{Aut}(H) \oplus \mathrm{Aut}(K)$.

Given this, the fundamental theorem shows that to compute the automorphism group of *G* it suffices to compute the automorphism groups of the Sylow *p*-subgroups separately (that is, all direct sums of cyclic subgroups, each with order a power of *p*). Fix a prime *p* and suppose the exponents e_i of the cyclic factors of the Sylow *p*-subgroup are arranged in increasing order:

$$e_1 \leq e_2 \leq \cdots \leq e_n$$

for some $n > 0$. One needs to find the automorphisms of

$$\mathbb{Z}_{p^{e_1}} \oplus \cdots \oplus \mathbb{Z}_{p^{e_n}}.$$

One special case is when $n = 1$, so that there is only one cyclic prime-power factor in the Sylow *p*-subgroup *P*. In this case the theory of automorphisms of a finite cyclic group can be used. Another special case is when *n* is arbitrary but $e_i = 1$ for $1 \leq i \leq n$. Here, one is considering *P* to be of the form

$$\mathbb{Z}_p \oplus \cdots \oplus \mathbb{Z}_p,$$

so elements of this subgroup can be viewed as comprising a vector space of dimension *n* over the finite field of *p* elements $\mathbb{F}_p$. The automorphisms of this subgroup are therefore given by the invertible linear transformations, so

$$\mathrm{Aut}(P) \cong \mathrm{GL}(n, \mathbb{F}_p),$$

where GL is the appropriate general linear group. This is easily shown to have order

$$|\mathrm{Aut}(P)| = (p^n - 1) \cdots (p^n - p^{n-1}).$$

In the most general case, where the e_i and *n* are arbitrary, the automorphism group is more difficult to determine. It is known, however, that if one defines

$$d_k = \max\{r | e_r = e_k\}$$

and

$$c_k = \min\{r | e_r = e_k\}$$

then one has in particular $d_k \geq k$, $c_k \leq k$, and

$$|\mathrm{Aut}(P)| = \left(\prod_{k=1}^{n} p^{d_k} - p^{k-1}\right) \left(\prod_{j=1}^{n} (p^{e_j})^{n-d_j}\right) \left(\prod_{i=1}^{n} (p^{e_i-1})^{n-c_i+1}\right).$$

One can check that this yields the orders in the previous examples as special cases (see [Hillar,Rhea]).

Infinite abelian groups

The simplest infinite abelian group is the infinite cyclic group **Z**. Any finitely generated abelian group A is isomorphic to the direct sum of r copies of **Z** and a finite abelian group, which in turn is decomposable into a direct sum of finitely many cyclic groups of primary orders. Even though the decomposition is not unique, the number r, called the **rank** of A, and the prime powers giving the orders of finite cyclic summands are uniquely determined.

By contrast, classification of general infinitely generated abelian groups is far from complete. Divisible groups, i.e. abelian groups A in which the equation $nx = a$ admits a solution $x \in A$ for any natural number n and element a of A, constitute one important class of infinite abelian groups that can be completely characterized. Every divisible group is isomorphic to a direct sum, with summands isomorphic to **Q** and Prüfer groups $\mathbf{Q}_p/\mathbf{Z}_p$ for various prime numbers p, and the cardinality of the set of summands of each type is uniquely determined.[2] Moreover, if a divisible group A is a subgroup of an abelian group G then A admits a direct complement: a subgroup C of G such that $G = A \oplus C$. Thus divisible groups are injective modules in the category of abelian groups, and conversely, every injective abelian group is divisible (Baer's criterion). An abelian group without non-zero divisible subgroups is called **reduced**.

Two important special classes of infinite abelian groups with diametrically opposite properties are *torsion groups* and *torsion-free groups*, examplified by the groups **Q/Z** (periodic) and **Q** (torsion-free).

Torsion groups

An abelian group is called **periodic** or **torsion** if every element has finite order. A direct sum of finite cyclic groups is periodic. Although the converse statement is not true in general, some special cases are known. The first and second Prüfer theorems state that if A is a periodic group and either it has **bounded exponent**, i.e. $nA = 0$ for some natural number n, or if A is countable and the p-heights of the elements of A are finite for each p, then A is isomorphic to a direct sum of finite cyclic groups.[3] The cardinality of the set of direct summands isomorphic to $\mathbf{Z}/p^m\mathbf{Z}$ in such a decomposition is an invariant of A. These theorems were later subsumed in the **Kulikov criterion**. In a different direction, Helmut Ulm found an extension of the second Prüfer theorem to countable abelian p-groups with elements of infinite height: those groups are completely classified by means of their Ulm invariants.

Torsion-free and mixed groups

An abelian group is called **torsion-free** if every non-zero element has infinite order. Several classes of torsion-free abelian groups have been extensively studied:

- Free abelian groups, i.e. arbitrary direct sums of **Z**
- Cotorsion and algebraically compact torsion-free groups such as the p-adic integers
- Slender groups

An abelian group that is neither periodic nor torsion-free is called **mixed**. If A is an abelian group and $T(A)$ is its torsion subgroup then the factor group $A/T(A)$ is torsion-free. However, in general the torsion subgroup is not a direct summand of A, so the torsion-free factor cannot be realized as a subgroup of A and A is *not* isomorphic to $T(A) \oplus A/T(A)$. Thus the theory of mixed groups involves more than simply combining the results about periodic and torsion-free groups.

Invariants and classification

One of the most basic invariants of an infinite abelian group A is its rank: the cardinality of the maximal linearly independent subset of A. Abelian groups of rank 0 are precisely the periodic groups, while torsion-free abelian groups of rank 1 are necessarily subgroups of **Q** and can be completely described. More generally, a torsion-free abelian group of finite rank r is a subgroup of $\mathbf{Q}^r$. On the other hand, the group of p-adic integers $\mathbf{Z}_p$ is a torsion-free abelian group of infinite **Z**-rank and the groups $\mathbf{Z}_p^n$ with different n are non-isomorphic, so this invariant does not even fully capture properties of some familiar groups.

The classification theorems for finitely generated, divisible, countable periodic, and rank 1 torsion-free abelian groups explained above were all obtained before 1950 and form a foundation of the classification of more general infinite abelian groups. Important technical tools used in classification of infinite abelian groups are pure and basic subgroups. Introduction of various invariants of torsion-free abelian groups has been one avenue of further progress. See the books by Irving Kaplansky, László Fuchs, Phillip Griffiths, and David Arnold, as well as the proceedings of the conferences on Abelian Group Theory published in Lecture Notes in Mathematics for more recent results.

Additive groups of rings

The additive group of a ring is an abelian group, but not all abelian groups are additive groups of rings. Some important topics in this area of study are:

- Tensor product
- Corner's results on countable torsion-free groups
- Shelah's work to remove cardinality restrictions

Relation to other mathematical topics

Many large abelian groups possess a natural topology, which turns them into topological groups.

The collection of all abelian groups, together with the homomorphisms between them, forms the category **Ab**, the prototype of an abelian category.

Nearly all well-known algebraic structures other than Boolean algebras, are undecidable. Hence it is surprising that Tarski's student Szmielew (1955) proved that the first order theory of abelian groups, unlike its nonabelian counterpart, is decidable. This decidability, plus the fundamental theorem of finite abelian groups described above, highlight some of the successes in abelian group theory, but there are still many areas of current research:

- Amongst torsion-free abelian groups of finite rank, only the finitely generated case and the rank 1 case are well understood;
- There are many unsolved problems in the theory of infinite-rank torsion-free abelian groups;
- While countable torsion abelian groups are well understood through simple presentations and Ulm invariants, the case of countable mixed groups is much less mature.
- Many mild extensions of the first order theory of abelian groups are known to be undecidable.
- Finite abelian groups remain a topic of research in computational group theory.

Moreover, abelian groups of infinite order lead, quite surprisingly, to deep questions about the set theory commonly assumed to underlie all of mathematics. Take the Whitehead problem: are all Whitehead groups of infinite order also free abelian groups? In the 1970s, Saharon Shelah proved that the Whitehead problem is:

- Undecidable in ZFC, the conventional axiomatic set theory from which nearly all of present day mathematics can be derived. The Whitehead problem is also the first question in ordinary mathematics proved undecidable in ZFC;
- Undecidable even if ZFC is augmented by taking the generalized continuum hypothesis as an axiom;
- Decidable if ZFC is augmented with the axiom of constructibility (see statements true in L).

A note on the typography

Among mathematical adjectives derived from the proper name of a mathematician, the word "abelian" is rare in that it is often spelled with a lowercase **a**, rather than an uppercase **A**, indicating how ubiquitous the concept is in modern mathematics.[4]

See also

- Abelianization
- Class field theory
- Commutator subgroup
- Elementary abelian group
- Pontryagin duality
- Pure injective module
- Pure projective module

Notes

[1] Jacobson (2009), p. 41

[2] For example, $\mathbf{Q}/\mathbf{Z} \cong \sum_p \mathbf{Q}_p/\mathbf{Z}_p$.

[3] Countability assumption in the second Prüfer theorem cannot be removed: the torsion subgroup of the direct product of the cyclic groups $\mathbf{Z}/p^m\mathbf{Z}$ for all natural m is not a direct sum of cyclic groups.

[4] Abel Prize Awarded: The Mathematicians' Nobel (http://www.maa.org/devlin/devlin_04_04.html)

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Group algebra

In mathematics, the **group algebra** is any of various constructions to assign to a locally compact group an operator algebra (or more generally a Banach algebra), such that representations of the algebra are related to representations of the group. As such, they are similar to the group ring associated to a discrete group.

Group algebras of topological groups: $C_c(G)$

For the purposes of functional analysis, and in particular of harmonic analysis, one wishes to carry over the group ring construction to topological groups G. In case G is a locally compact Hausdorff group, G carries an essentially unique left-invariant countably additive Borel measure μ called Haar measure. Using the Haar measure, one can define a convolution operation on the space $C_c(G)$ of complex-valued functions on G with compact support; $C_c(G)$ can then be given any of various norms and the completion will be a group algebra.

To define the convolution operation, let f and g be two functions in $C_c(G)$. For t in G, define

$$[f * g](t) = \int_G f(s) g(s^{-1} t)\, d\mu(s).$$

The fact that $f * g$ is continuous is immediate from the dominated convergence theorem. Also

$$\operatorname{Support}(f * g) \subseteq \operatorname{Support}(f) \cdot \operatorname{Support}(g)$$

$C_c(G)$ also has a natural involution defined by:

$$f^*(s) = \overline{f(s^{-1})} \Delta(s^{-1})$$

where Δ is the modular function on G. With this involution, it is a *-algebra.

Theorem. If $C_c(G)$ is given the norm

$$\|f\|_1 := \int_G |f(s)| d\mu(s),$$ it becomes is an involutive normed algebra with an approximate identity.

The approximate identity can be indexed on a neighborhood basis of the identity consisting of compact sets. Indeed if V is a compact neighborhood of the identity, let f_V be a non-negative continuous function supported in V such that

$$\int_V f_V(g)\, d\mu(g) = 1.$$

Then $\{f_V\}_V$ is an approximate identity. A group algebra can only have an identity, as opposed to just approximate identity, if and only if the topology on the group is the discrete topology.

Note that for discrete groups, $C_c(G)$ is the same thing as the complex group ring $\mathbf{C}G$.

The importance of the group algebra is that it captures the unitary representation theory of G as shown in the following

Theorem. Let G be a locally compact group. If U is a strongly continuous unitary representation of G on a Hilbert space H, then

$$\pi_U(f) = \int_G f(g) U(g)\, d\mu(g)$$

is a non-degenerate bounded *-representation of the normed algebra $C_c(G)$. The map

$$U \mapsto \pi_U$$

is a bijection between the set of strongly continuous unitary representations of G and non-degenerate bounded *-representations of $C_c(G)$. This bijection respects unitary equivalence and strong containment. In particular, π_U is irreducible if and only if U is irreducible.

Non-degeneracy of a representation π of $C_c(G)$ on a Hilbert space H_π means that

$$\{\pi(f)\xi : f \in \mathrm{C}_c(G), \xi \in H_\pi\}$$

is dense in H_π.

The convolution algebra $L^1(G)$

It is a standard theorem of measure theory that the completion of $C_c(G)$ in the $L^1(G)$ norm is isomorphic to the space $L^1(G)$ of equivalence classes of functions which are integrable with respect to the Haar measure, where, as usual, two functions are regarded as equivalent if and only if they differ on a set of Haar measure zero.

Theorem. $L^1(G)$ is a Banach *-algebra with the convolution product and involution defined above and with the L^1 norm. $L^1(G)$ also has a bounded approximate identity.

The group C*-algebra C*(G)

Let **C**[*G*] be the group ring of a discrete group *G*.

For a locally compact group *G*, the group C*-algebra *C**(*G*) of *G* is defined to be the C*-enveloping algebra of $L^1(G)$, i.e. the completion of $C_c(G)$ with respect to the largest C*-norm:

$$\|f\|_{C^*} := \sup_\pi \|\pi(f)\|,$$

where π ranges over all non-degenerate *-representations of $C_c(G)$ on Hilbert spaces. When *G* is discrete, it follows from the triangle inequality that, for any such π, $\pi(f) \le \|f\|_1$. So the norm is well-defined.

It follows from the definition that *C**(*G*) has the following universal property: any *-homomorphism from **C**[*G*] to some **B**($\mathcal{H}$) (the *C**-algebra of bounded operators on some Hilbert space $\mathcal{H}$) factors through the inclusion map **C**[*G*] $\hookrightarrow$ $C^*_{\max}(G)$.

The reduced group C*-algebra $C^*_r(G)$

The reduced group C*-algebra $C^*_r(G)$ is the completion of $C_c(G)$ with respect to the norm

$$\|f\|_{C^*_r} := \sup\{\|f * g\|_2 : \|g\|_2 = 1\},$$

where

$$\|f\|_2 = \sqrt{\int_G |f|^2 d\mu}$$

is the L^2 norm. Since the completion of $C_c(G)$ with regard to the L^2 norm is a Hilbert space, the C^*_r norm is the norm of the bounded operator "convolution by *f*" acting on $L^2(G)$ and thus a C*-norm.

Equivalently, $C^*_r(G)$ is the C*-algebra generated by the image of the left regular representation on $l^2(G)$.

In general, $C^*_r(G)$ is a quotient of $C^*(G)$. The reduced group C*-algebra is isomorphic to the non-reduced group C*-algebra defined above if and only if *G* is amenable.

von Neumann algebras associated to groups

The group von Neumann algebra W*(*G*) of *G* is the enveloping von Neumann algebra of C*(*G*).

For a discrete group *G*, we can consider the Hilbert space $l^2(G)$ for which *G* is an orthonormal basis. Since *G* operates on $l^2(G)$ by permuting the basis vectors, we can identify the complex group ring **C***G* with a subalgebra of the algebra of bounded operators on $l^2(G)$. The weak closure of this subalgebra, *NG*, is a von Neumann algebra.

The center of *NG* can be described in terms of those elements of *G* whose conjugacy class is finite. In particular, if the identity element of *G* is the only group element with that property (that is, *G* has the infinite conjugacy class property), the center of *NG* consists only of complex multiples of the identity.

NG is isomorphic to the hyperfinite type II_1 factor if and only if *G* is countable, amenable, and has the infinite conjugacy class property.

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[1] http://eom.springer.de/G/g045230.htm

Cayley's theorem

In group theory, **Cayley's theorem**, named in honor of Arthur Cayley, states that every group G is isomorphic to a subgroup of the symmetric group on G.[1] This can be understood as an example of the group action of G on the elements of G.[2]

A permutation of a set G is any bijective function taking G onto G; and the set of all such functions forms a group under function composition, called *the symmetric group on* G, and written as Sym(G).[3]

Cayley's theorem puts all groups on the same footing, by considering any group (including infinite groups such as ($\boldsymbol{R}$,+)) as a permutation group of some underlying set. Thus, theorems which are true for permutation groups are true for groups in general.

History

Although Burnside[4] attributes the theorem to Jordan,[5] Eric Nummela[6] nonetheless argues that the standard name—"Cayley's Theorem"—is in fact appropriate. Cayley, in his original 1854 paper,[7] showed that the correspondence in the theorem is one-to-one, but he failed to explicitly show it was a homomorphism (and thus an isomorphism). However, Nummela notes that Cayley made this result known to the mathematical community at the time, thus predating Jordan by 16 years or so.

Proof of the theorem

Where g is any element of G, consider the function $f_g : G \to G$, defined by $f_g(x) = g*x$. By the existence of inverses, this function has a two-sided inverse, $f_{g^{-1}}$. So multiplication by g acts as a bijective function. Thus, f_g is a permutation of G, and so is a member of Sym(G).

The set $K = \{f_g : g \text{ in } G\}$ is a subgroup of Sym(G) which is isomorphic to G. The fastest way to establish this is to consider the function $T : G \to \text{Sym}(G)$ with $T(g) = f_g$ for every g in G. T is a group homomorphism because (using "•" for composition in Sym(G)):

$$(f_g \bullet f_h)(x) = f_g(f_h(x)) = f_g(h*x) = g*(h*x) = (g*h)*x = f_{(g*h)}(x),$$

for all x in G, and hence:

$$T(g) \bullet T(h) = f_g \bullet f_h = f_{(g*h)} = T(g*h).$$

The homomorphism T is also injective since $T(g) = \text{id}_G$ (the identity element of Sym(G)) implies that $g*x = x$ for all x in G, and taking x to be the identity element e of G yields $g = g*e = e$. Alternatively, T is also injective since, if $g*x=g'*x$ implies $g=g'$ (by post-multiplying with the inverse of x, which exists because G is a group).

Thus *G* is isomorphic to the image of *T*, which is the subgroup *K*.

T is sometimes called the *regular representation of G*.

Alternative setting of proof

An alternative setting uses the language of group actions. We consider the group G as a G-set, which can be shown to have permutation representation, say ϕ .

Firstly, suppose $G = G/H$ with $H = \{e\}$. Then the group action is $g.e$ by classification of G-orbits (also known as the orbit-stabilizer theorem).

Now, the representation is faithful if ϕ is injective, that is, if the kernel of ϕ is trivial. Suppose g ∈ ker ϕ Then, $g = g.e = \phi(g).e$ by the equivalence of the permutation representation and the group action. But since g ∈ ker ϕ , $\phi(g) = e$ and thus ker ϕ is trivial. Then im $\phi < G$ and thus the result follows by use of the first isomorphism theorem.

Remarks on the regular group representation

The identity group element corresponds to the identity permutation. All other group elements correspond to a permutation that does not leave any element unchanged. Since this also applies for powers of a group element, lower than the order of that element, each element corresponds to a permutation which consists of cycles which are of the same length: this length is the order of that element. The elements in each cycle form a left coset of the subgroup generated by the element.

Examples of the regular group representation

Z_2 = {0,1} with addition modulo 2; group element 0 corresponds to the identity permutation e, group element 1 to permutation (12).

Z_3 = {0,1,2} with addition modulo 3; group element 0 corresponds to the identity permutation e, group element 1 to permutation (123), and group element 2 to permutation (132). E.g. 1 + 1 = 2 corresponds to (123)(123)=(132).

Z_4 = {0,1,2,3} with addition modulo 4; the elements correspond to e, (1234), (13)(24), (1432).

The elements of Klein four-group {e, a, b, c} correspond to e, (12)(34), (13)(24), and (14)(23).

S_3 (dihedral group of order 6) is the group of all permutations of 3 objects, but also a permutation group of the 6 group elements:

*	*e*	*a*	*b*	*c*	*d*	*f*	permutation
e	*e*	*a*	*b*	*c*	*d*	*f*	*e*
a	*a*	*e*	*d*	*f*	*b*	*c*	(12)(35)(46)
b	*b*	*f*	*e*	*d*	*c*	*a*	(13)(26)(45)
c	*c*	*d*	*f*	*e*	*a*	*b*	(14)(25)(36)
d	*d*	*c*	*a*	*b*	*f*	*e*	(156)(243)
f	*f*	*b*	*c*	*a*	*e*	*d*	(165)(234)

See also

- Containment order, a similar result in order theory
- Frucht's theorem, every group is the automorphism group of a graph
- Yoneda lemma, an analogue of Cayley's theorem in category theory

Notes

[1] Jacobson (2009), p. 38.
[2] Jacobson (2009), p. 72, ex. 1.
[3] Jacobson (2009), p. 31.
[4] Burnside, William (1911), *Theory of Groups of Finite Order* (2 ed.), Cambridge, ISBN 0486495752
[5] Jordan, Camille (1870), *Traite des substitutions et des equations algebriques*, Paris: Gauther-Villars
[6] Nummela, Eric (1980), "Cayley's Theorem for Topological Groups" (http://jstor.org/stable/2321608), *American Mathematical Monthly* (Mathematical Association of America) **87** (3): 202–203, doi:10.2307/2321608,
[7] Cayley, Arthur (1854), "On the theory of groups as depending on the symbolic equation $\theta^n=1$", *Phil. Mag.* **7** (4): 40–47

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Special Algebras, Operator Algebra and Quantum Algebra

Lie algebra

In mathematics, a **Lie algebra** (pronounced /ˈliː/ ("lee"), not /ˈlaɪ/ ("lye")) is an algebraic structure whose main use is in studying geometric objects such as Lie groups and differentiable manifolds. Lie algebras were introduced to study the concept of infinitesimal transformations. The term "Lie algebra" (after Sophus Lie) was introduced by Hermann Weyl in the 1930s. In older texts, the name "**infinitesimal group**" is used.

Definition and first properties

A Lie algebra is a vector space $\mathfrak{g}$ over some field F together with a binary operation [·, ·]

$$[\cdot,\cdot]:\mathfrak{g}\times\mathfrak{g}\to\mathfrak{g}$$

called the **Lie bracket**, which satisfies the following axioms:

- Bilinearity:

$$[ax+by,z]=a[x,z]+b[y,z],\quad [z,ax+by]=a[z,x]+b[z,y]$$

  for all scalars a, b in F and all elements x, y, z in $\mathfrak{g}$.
- Alternating on $\mathfrak{g}$:

$$[x,x]=0$$

  for all x in $\mathfrak{g}$. This implies anticommutativity, or skew-symmetry (in fact the conditions are equivalent for any Lie algebra over any field whose characteristic is not 2):

$$[x,y]=-[y,x]$$

  for all elements x, y in $\mathfrak{g}$.
- The Jacobi identity:

$$[x,[y,z]]+[y,[z,x]]+[z,[x,y]]=0$$

  for all x, y, z in $\mathfrak{g}$.

For any associative algebra A with multiplication $*$, one can construct a Lie algebra $L(A)$. As a vector space, $L(A)$ is the same as A. The Lie bracket of two elements of $L(A)$ is defined to be their commutator in A:

$$[a,b]=a*b-b*a.$$

The associativity of the multiplication * in A implies the Jacobi identity of the commutator in $L(A)$. In particular, the associative algebra of $n \times n$ matrices over a field F gives rise to the general linear Lie algebra $\mathfrak{gl}_n(F).$ The associative algebra A is called an **enveloping algebra** of the Lie algebra $L(A)$. It is known that every Lie algebra can be embedded into one that arises from an associative algebra in this fashion. See universal enveloping algebra.

Homomorphisms, subalgebras, and ideals

The Lie bracket is not an associative operation in general, meaning that $[[x,y],z]$ need not equal $[x,[y,z]]$. Nonetheless, much of the terminology that was developed in the theory of associative rings or associative algebras is commonly applied to Lie algebras. A subspace $\mathfrak{h} \subseteq \mathfrak{g}$ that is closed under the Lie bracket is called a **Lie subalgebra**. If a subspace $I \subseteq \mathfrak{g}$ satisfies a stronger condition that

$$[\mathfrak{g}, I] \subseteq I,$$

then I is called an **ideal** in the Lie algebra $\mathfrak{g}$.[1] A Lie algebra in which the commutator is not identically zero and which has no proper ideals is called **simple**. A **homomorphism** between two Lie algebras (over the same ground field) is a linear map that is compatible with the commutators:

$$f : \mathfrak{g} \to \mathfrak{g}', \quad f([x,y]) = [f(x), f(y)],$$

for all elements x and y in $\mathfrak{g}$. As in the theory of associative rings, ideals are precisely the kernels of homomorphisms, given a Lie algebra $\mathfrak{g}$ and an ideal I in it, one constructs the **factor algebra** $\mathfrak{g}/I$, and the first isomorphism theorem holds for Lie algebras. Given two Lie algebras $\mathfrak{g}$ and $\mathfrak{g}'$, their direct sum is the vector space $\mathfrak{g} \oplus \mathfrak{g}'$ consisting of the pairs $(x, x'),\ x \in \mathfrak{g}, x' \in \mathfrak{g}'$, with the operation

$$[(x,x'),(y,y')] = ([x,y],[x',y']), \quad x, y \in \mathfrak{g},\ x', y' \in \mathfrak{g}'.$$

Examples

- Any vector space V endowed with the identically zero Lie bracket becomes a Lie algebra. Such Lie algebras are called abelian, cf. below. Any one-dimensional Lie algebra over a field is abelian, by the antisymmetry of the Lie bracket.
- The three-dimensional Euclidean space $\mathbf{R}^3$ with the Lie bracket given by the cross product of vectors becomes a three-dimensional Lie algebra.
- The Heisenberg algebra is a three-dimensional Lie algebra with generators (see also the definition at Generating set):

$$x = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad y = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad z = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

  whose commutation relations are

$$[x,y] = z, \quad [x,z] = 0, \quad [y,z] = 0.$$

  It is explicitly exhibited as the space of 3×3 strictly upper-triangular matrices.
- The subspace of the general linear Lie algebra $\mathfrak{gl}_n(F)$ consisting of matrices of trace zero is a subalgebra,[2] the *special linear Lie algebra*, denoted $\mathfrak{sl}_n(F)$.
- Any Lie group G defines an associated real Lie algebra $\mathfrak{g} = \mathrm{Lie}(G)$. The definition in general is somewhat technical, but in the case of real matrix groups, it can be formulated via the exponential map, or the matrix exponent. The Lie algebra $\mathfrak{g}$ consists of those matrices X for which

$$\exp(tX) \in G$$

  for all real numbers t. The Lie bracket of $\mathfrak{g}$ is given by the commutator of matrices. As a concrete example, consider the special linear group SL(n,**R**), consisting of all $n \times n$ matrices with real entries and determinant 1. This is a matrix Lie group, and its Lie algebra consists of all $n \times n$ matrices with real entries and trace 0.
- The real vector space of all $n \times n$ skew-hermitian matrices is closed under the commutator and forms a real Lie algebra denoted $\mathfrak{u}(n)$. This is the Lie algebra of the unitary group $U(n)$.
- An important class of infinite-dimensional real Lie algebras arises in differential topology. The space of smooth vector fields on a differentiable manifold M forms a Lie algebra, where the Lie bracket is defined to be the

commutator of vector fields. One way of expressing the Lie bracket is through the formalism of Lie derivatives, which identifies a vector field X with a first order partial differential operator L_X acting on smooth functions by letting $L_X(f)$ be the directional derivative of the function f in the direction of X. The Lie bracket $[X,Y]$ of two vector fields is the vector field defined through its action on functions by the formula:

$$L_{[X,Y]}f = L_X(L_Y f) - L_Y(L_X f).$$

This Lie algebra is related to the pseudogroup of diffeomorphisms of M.

- The commutation relations between the x, y, and z components of the angular momentum operator in quantum mechanics form a representation of a complex three-dimensional Lie algebra, which is the complexification of the Lie algebra $so(3)$ of the three-dimensional rotation group:

$$[L_x, L_y] = i\hbar L_z$$
$$[L_y, L_z] = i\hbar L_x$$
$$[L_z, L_x] = i\hbar L_y$$

- Kac–Moody algebra is an example of an infinite-dimensional Lie algebra.

Structure theory and classification

Every finite-dimensional real or complex Lie algebra has a faithful representation by matrices (Ado's theorem). Lie's fundamental theorems describe a relation between Lie groups and Lie algebras. In particular, any Lie group gives rise to a canonically determined Lie algebra (concretely, the tangent space at the identity), and conversely, for any Lie algebra there is a corresponding connected Lie group (Lie's third theorem). This Lie group is not determined uniquely, however, any two connected Lie groups with the same Lie algebra are *locally isomorphic*, and in particular, have the same universal cover. For instance, the special orthogonal group SO(3) and the special unitary group SU(2) give rise to the same Lie algebra, which is isomorphic to $\mathbf{R}^3$ with the cross-product, and SU(2) is a simply-connected twofold cover of SO(3). Real and complex Lie algebras can be classified to some extent, and this is often an important step toward the classification of Lie groups.

Abelian, nilpotent, and solvable

Analogously to abelian, nilpotent, and solvable groups, defined in terms of the derived subgroups, one can define abelian, nilpotent, and solvable Lie algebras.

A Lie algebra $\mathfrak{g}$ is **abelian** if the Lie bracket vanishes, i.e. $[x,y] = 0$, for all x and y in $\mathfrak{g}$. Abelian Lie algebras correspond to commutative (or abelian) connected Lie groups such as vector spaces K^n or tori T^n, and are all of the form $\mathfrak{k}^n$, meaning an n-dimensional vector space with the trivial Lie bracket.

A more general class of Lie algebras is defined by the vanishing of all commutators of given length. A Lie algebra $\mathfrak{g}$ is **nilpotent** if the lower central series

$$\mathfrak{g} > [\mathfrak{g},\mathfrak{g}] > [[\mathfrak{g},\mathfrak{g}],\mathfrak{g}] > [[[\mathfrak{g},\mathfrak{g}],\mathfrak{g}],\mathfrak{g}] > \cdots$$

becomes zero eventually. By Engel's theorem, a Lie algebra is nilpotent if and only if for every u in $\mathfrak{g}$ the adjoint endomorphism

$$\mathrm{ad}(u) : \mathfrak{g} \to \mathfrak{g}, \quad \mathrm{ad}(u)v = [u,v]$$

is nilpotent.

More generally still, a Lie algebra $\mathfrak{g}$ is said to be **solvable** if the derived series:

$$\mathfrak{g} > [\mathfrak{g},\mathfrak{g}] > [[\mathfrak{g},\mathfrak{g}],[\mathfrak{g},\mathfrak{g}]] > [[[\mathfrak{g},\mathfrak{g}],[\mathfrak{g},\mathfrak{g}]],[[\mathfrak{g},\mathfrak{g}],[\mathfrak{g},\mathfrak{g}]]] > \cdots$$

becomes zero eventually.

Every finite-dimensional Lie algebra has a unique maximal solvable ideal, called its radical. Under the Lie correspondence, nilpotent (respectively, solvable) connected Lie groups correspond to nilpotent (respectively, solvable) Lie algebras.

Simple and semisimple

A Lie algebra is "simple" if it has no non-trivial ideals and is not abelian. A Lie algebra $\mathfrak{g}$ is called **semisimple** if its radical is zero. Equivalently, $\mathfrak{g}$ is semisimple if it does not contain any non-zero abelian ideals. In particular, a simple Lie algebra is semisimple. Conversely, it can be proven that any semisimple Lie algebra is the direct sum of its minimal ideals, which are canonically determined simple Lie algebras.

The concept of semisimplicity for Lie algebras is closely related with the complete reducibility of their representations. When the ground field *F* has characteristic zero, semisimplicity of a Lie algebra $\mathfrak{g}$ over *F* is equivalent to the complete reducibility of all finite-dimensional representations of $\mathfrak{g}$. An early proof of this statement proceeded via connection with compact groups (Weyl's unitary trick), but later entirely algebraic proofs were found.

Classification

In many ways, the classes of semisimple and solvable Lie algebras are at the opposite ends of the full spectrum of the Lie algebras. The Levi decomposition expresses an arbitrary Lie algebra as a semidirect sum of its solvable radical and a semisimple Lie algebra, almost in a canonical way. Semisimple Lie algebras over an algebraically closed field have been completely classified through their root systems. The classification of solvable Lie algebras is a 'wild' problem, and cannot be accomplished in general.

Cartan's criterion gives conditions for a Lie algebra to be nilpotent, solvable, or semisimple. It is based on the notion of the Killing form, a symmetric bilinear form on $\mathfrak{g}$ defined by the formula

$$K(u,v) = \operatorname{tr}(\operatorname{ad}(u)\operatorname{ad}(v)),$$

where tr denotes the trace of a linear operator. A Lie algebra $\mathfrak{g}$ is semisimple if and only if the Killing form is nondegenerate. A Lie algebra $\mathfrak{g}$ is solvable if and only if $K(\mathfrak{g}, [\mathfrak{g}, \mathfrak{g}]) = 0.$

Relation to Lie groups

Although Lie algebras are often studied in their own right, historically they arose as a means to study Lie groups. Given a Lie group, a Lie algebra can be associated to it either by endowing the tangent space to the identity with the differential of the adjoint map, or by considering the left-invariant vector fields as mentioned in the examples. This association is functorial, meaning that homomorphisms of Lie groups lift to homomorphisms of Lie algebras, and various properties are satisfied by this lifting: it commutes with composition, it maps Lie subgroups, kernels, quotients and cokernels of Lie groups to subalgebras, kernels, quotients and cokernels of Lie algebras, respectively.

The functor which takes each Lie group to its Lie algebra and each homomorphism to its differential is a faithful and exact functor. This functor is not invertible; different Lie groups may have the same Lie algebra, for example SO(3) and SU(2) have isomorphic Lie algebras. Even worse, some Lie algebras need not have *any* associated Lie group. Nevertheless, when the Lie algebra is finite-dimensional, there is always at least one Lie group whose Lie algebra is the one under discussion, and a preferred Lie group can be chosen. Any finite-dimensional connected Lie group has a universal cover. This group can be constructed as the image of the Lie algebra under the exponential map. More generally, we have that the Lie algebra is homeomorphic to a neighborhood of the identity. But globally, if the Lie group is compact, the exponential will not be injective, and if the Lie group is not connected, simply connected or compact, the exponential map need not be surjective.

If the Lie algebra is infinite-dimensional, the issue is more subtle. In many instances, the exponential map is not even locally a homeomorphism (for example, in Diff($\mathbf{S}^1$), one may find diffeomorphisms arbitrarily close to the identity which are not in the image of exp). Furthermore, some infinite-dimensional Lie algebras are not the Lie algebra of any group.

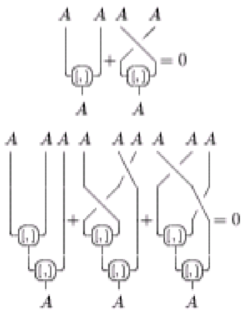
The correspondence between Lie algebras and Lie groups is used in several ways, including in the classification of Lie groups and the related matter of the representation theory of Lie groups. Every representation of a Lie algebra lifts uniquely to a representation of the corresponding connected, simply connected Lie group, and conversely every representation of any Lie group induces a representation of the group's Lie algebra; the representations are in one to one correspondence. Therefore, knowing the representations of a Lie algebra settles the question of representations of the group. As for classification, it can be shown that any connected Lie group with a given Lie algebra is isomorphic to the universal cover mod a discrete central subgroup. So classifying Lie groups becomes simply a matter of counting the discrete subgroups of the center, once the classification of Lie algebras is known (solved by Cartan et al. in the semisimple case).

Category theoretic definition

Using the language of category theory, a **Lie algebra** can be defined as an object A in **Vec**, the category of vector spaces together with a morphism [.,.]: $A \otimes A \rightarrow A$, where $\otimes$ refers to the monoidal product of **Vec**, such that

- $[\cdot,\cdot] \circ (\mathrm{id} + \tau_{A,A}) = 0$
- $[\cdot,\cdot] \circ ([\cdot,\cdot] \otimes \mathrm{id}) \circ (\mathrm{id} + \sigma + \sigma^2) = 0$

where $\tau\,(a \otimes b) := b \otimes a$ and σ is the cyclic permutation braiding $(\mathrm{id} \otimes \tau_{A,A}) \circ (\tau_{A,A} \otimes \mathrm{id})$. In diagrammatic form:



Notes

[1] Due to the anticommutativity of the commutator, the notions of a left and right ideal in a Lie algebra coincide.
[2] Humphreys p.2

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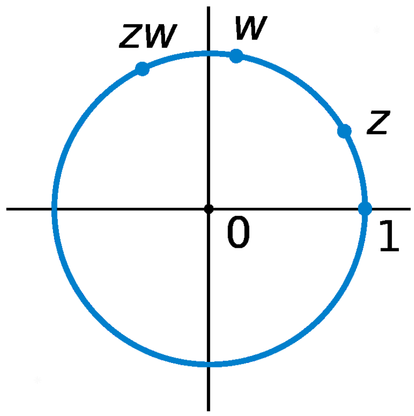
Lie group

In mathematics, a **Lie group** (pronounced /ˈliː/: similar to "Lee") is a group which is also a differentiable manifold, with the property that the group operations are compatible with the smooth structure. Lie groups are named after Sophus Lie, who laid the foundations of the theory of continuous transformation groups.

Lie groups represent the best-developed theory of continuous symmetry of mathematical objects and structures, which makes them indispensable tools for many parts of contemporary mathematics, as well as for modern theoretical physics. They provide a natural framework for analysing the continuous symmetries of differential equations (Differential Galois theory), in much the same way as permutation groups are used in Galois theory for analysing the discrete symmetries of algebraic equations. An extension of Galois theory to the case of continuous symmetry groups was one of Lie's principal motivations.

Overview

Lie groups are smooth manifolds and, therefore, can be studied using differential calculus, in contrast with the case of more general topological groups. One of the key ideas in the theory of Lie groups, from Sophus Lie, is to replace the *global* object, the group, with its *local* or linearized version, which Lie himself called its "infinitesimal group" and which has since become known as its Lie algebra.



The circle of center 0 and radius 1 in the complex plane is a Lie group with complex multiplication.

Lie groups play an enormous role in modern geometry, on several different levels. Felix Klein argued in his Erlangen program that one can consider various "geometries" by specifying an appropriate transformation group that leaves certain geometric properties invariant. Thus Euclidean geometry corresponds to the choice of the group E(3) of distance-preserving transformations of the Euclidean space $\mathbf{R}^3$, conformal geometry corresponds to enlarging the group to the conformal group, whereas in projective geometry one is interested in the properties invariant under the projective group. This idea later led to the notion of a G-structure, where G is a Lie group of "local" symmetries of a manifold. On a "global" level, whenever a Lie group acts on a geometric object, such as a Riemannian or a symplectic manifold, this action provides a measure of rigidity and yields a rich algebraic structure. The presence of continuous symmetries expressed via a Lie group action on a manifold places strong constraints on its geometry and facilitates analysis on the manifold. Linear actions of Lie groups are especially important, and are studied in representation theory.

In the 1940s–1950s, Ellis Kolchin, Armand Borel and Claude Chevalley realised that many foundational results concerning Lie groups can be developed completely algebraically, giving rise to the theory of algebraic groups defined over an arbitrary field. This insight opened new possibilities in pure algebra, by providing a uniform construction for most finite simple groups, as well as in algebraic geometry. The theory of automorphic forms, an important branch of modern number theory, deals extensively with analogues of Lie groups over adele rings; p-adic Lie groups play an important role, via their connections with Galois representations in number theory.

Definitions and examples

A **real Lie group** is a group which is also a finite-dimensional real smooth manifold, and in which the group operations of multiplication and inversion are smooth maps. Smoothness of the group multiplication

$$\mu : G \times G \to G \quad \mu(x,y) = xy$$

means that μ is a smooth mapping of the product manifold *G*×*G* into *G*. These two requirements can be combined to the single requirement that the mapping

$$(x,y) \mapsto x^{-1}y$$

be a smooth mapping of the product manifold into *G*.

First examples

- The 2×2 real invertible matrices form a group under multiplication, denoted by $GL_2(\mathbf{R})$:

$$GL_2(\mathbb{R}) = \left\{ A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} : \det A = ad - bc \neq 0 \right\}.$$

  This is a four-dimensional noncompact real Lie group. This group is disconnected; it has two connected components corresponding to the positive and negative values of the determinant.

- The rotation matrices form a subgroup of $GL_2(\mathbf{R})$, denoted by $SO_2(\mathbf{R})$. It is a Lie group in its own right: specifically, a one-dimensional compact connected Lie group which is diffeomorphic to the circle. Using the rotation angle φ as a parameter, this group can be parametrized as follows:

$$SO_2(\mathbb{R}) = \left\{ \begin{pmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix} : \varphi \in \mathbb{R}/2\pi\mathbb{Z} \right\}.$$

  Addition of the angles corresponds to multiplication of the elements of $SO_2(\mathbf{R})$, and taking the opposite angle corresponds to inversion. Thus both multiplication and inversion are differentiable maps.

- The orthogonal group also forms an interesting example of a Lie group.

All of the previous examples of Lie groups fall within the class of classical groups

Related concepts

A **complex Lie group** is defined in the same way using complex manifolds rather than real ones (example: $SL_2(\mathbf{C})$), and similarly one can define a ***p*-adic Lie group** over the *p*-adic numbers. Hilbert's fifth problem asked whether replacing differentiable manifolds with topological or analytic ones can yield new examples. The answer to this question turned out to be negative: in 1952, Gleason, Montgomery and Zippin showed that if *G* is a topological manifold with continuous group operations, then there exists exactly one analytic structure on *G* which turns it into a Lie group (see also Hilbert–Smith conjecture). If the underlying manifold is allowed to be infinite dimensional (for example, a Hilbert manifold) then one arrives at the notion of an infinite-dimensional Lie group. It is possible to define analogues of many Lie groups over finite fields, and these give most of the examples of finite simple groups.

The language of category theory provides a concise definition for Lie groups: a Lie group is a group object in the category of smooth manifolds. This is important, because it allows generalization of the notion of a Lie group to Lie supergroups.

More examples of Lie groups

Lie groups occur in abundance throughout mathematics and physics. Matrix groups or algebraic groups are (roughly) groups of matrices (for example, orthogonal and symplectic groups), and these give most of the more common examples of Lie groups.

Examples

- Euclidean space $\mathbf{R}^n$ with ordinary vector addition as the group operation becomes an n-dimensional noncompact abelian Lie group.
- The circle group $\mathbf{S}^1$ consisting of angles mod 2π under addition or, alternately, the complex numbers with absolute value 1 under multiplication. This is a one-dimensional compact connected abelian Lie group.
- The group $\mathrm{GL}_n(\mathbf{R})$ of invertible matrices (under matrix multiplication) is a Lie group of dimension n^2, called the general linear group. It has a closed connected subgroup $\mathrm{SL}_n(\mathbf{R})$, the special linear group, consisting of matrices of determinant 1 which is also a Lie group.
- The orthogonal group $\mathrm{O}_n(\mathbf{R})$, consisting of all $n \times n$ orthogonal matrices with real entries is an $n(n - 1)/2$-dimensional Lie group. This group is disconnected, but it has a connected subgroup $\mathrm{SO}_n(\mathbf{R})$ of the same dimension consisting of orthogonal matrices of determinant 1, called the special orthogonal group (for $n = 3$, the rotation group).
- The Euclidean group $\mathrm{E}_n(\mathbf{R})$ is the Lie group of all Euclidean motions, i.e., isometric affine maps, of n-dimensional Euclidean space $\mathbf{R}^n$.
- The unitary group $\mathrm{U}(n)$ consisting of $n \times n$ unitary matrices (with complex entries) is a compact connected Lie group of dimension n^2. Unitary matrices of determinant 1 form a closed connected subgroup of dimension $n^2 - 1$ denoted $\mathrm{SU}(n)$, the special unitary group.
- Spin groups are double covers of the special orthogonal groups, used for studying fermions in quantum field theory (among other things).
- The symplectic group $\mathrm{Sp}_{2n}(\mathbf{R})$ consists of all $2n \times 2n$ matrices preserving a nondegenerate skew-symmetric bilinear form on $\mathbf{R}^{2n}$ (the *symplectic form*). It is a connected Lie group of dimension $2n^2 + n$. The fundamental group of the symplectic group is $\mathbf{Z}$ and this fact is related to the theory of Maslov index.
- The 3-sphere $\mathbf{S}^3$ forms a Lie group by identification with the set of quaternions of unit norm, called versors. The only other spheres that admit the structure of a Lie group are the 0-sphere $\mathbf{S}^0$ (real numbers with absolute value 1) and the circle $\mathbf{S}^1$ (complex numbers with absolute value 1). For example, for even $n > 1$, S^n is not a Lie group because it does not admit a nonvanishing vector field and so *a fortiori* cannot be parallelizable as a differentiable manifold. Of the spheres only S^0, S^1, S^3, and S^7 are parallelizable. The latter carries the structure of a Lie quasigroup (a nonassociative group), which can be identified with the set of unit octonions.
- The group of upper triangular n by n matrices is a solvable Lie group of dimension $n(n + 1)/2$.
- The Lorentz group and the Poincare group are the groups of linear and affine isometries of the Minkowski space (interpreted as the spacetime of the special relativity). They are Lie groups of dimensions 6 and 10.
- The Heisenberg group is a connected nilpotent Lie group of dimension 3, playing a key role in quantum mechanics.
- The group U(1)×SU(2)×SU(3) is a Lie group of dimension 1+3+8=12 that is the gauge group of the Standard Model in particle physics. The dimensions of the factors correspond to the 1 photon + 3 vector bosons + 8 gluons of the standard model.
- The (3-dimensional) metaplectic group is a double cover of $\mathrm{SL}_2(\mathbf{R})$ playing an important role in the theory of modular forms. It is a connected Lie group that cannot be faithfully represented by matrices of finite size, i.e., a nonlinear group.
- The exceptional Lie groups of types G_2, F_4, E_6, E_7, E_8 have dimensions 14, 52, 78, 133, and 248. There is also a group $\mathrm{E}_{7½}$ of dimension 190.

Constructions

There are several standard ways to form new Lie groups from old ones:

- The product of two Lie groups is a Lie group.
- Any topologically closed subgroup of a Lie group is a Lie group. This is known as Cartan's theorem.
- The quotient of a Lie group by a closed normal subgroup is a Lie group.
- The universal cover of a connected Lie group is a Lie group. For example, the group **R** is the universal cover of the circle group $\mathbf{S}^1$. In fact any covering of a differentiable manifold is also a differentiable manifold, but by specifying *universal* cover, one guarantees a group structure (compatible with its other structures).

Related notions

Some examples of groups that are *not* Lie groups (except in the trivial sense that any group can be viewed as a 0-dimensional Lie group, with the discrete topology), are:

- Infinite dimensional groups, such as the additive group of an infinite dimensional real vector space. These are not Lie groups as they are not *finite dimensional* manifolds
- Some totally disconnected groups, such as the Galois group of an infinite extension of fields, or the additive group of the *p*-adic numbers. These are not Lie groups because their underlying spaces are not real manifolds. (Some of these groups are "*p*-adic Lie groups"). In general, only topological groups having similar local properties to $\mathbf{R}^\mathbf{n}$ for some positive integer *n* can be Lie groups (of course they must also have a differentiable structure)

Early history

According to the most authoritative source on the early history of Lie groups (Hawkins, p. 1), Sophus Lie himself considered the winter of 1873–1874 as the birth date of his theory of continuous groups. Hawkins, however, suggests that it was "Lie's prodigious research activity during the four-year period from the fall of 1869 to the fall of 1873" that led to the theory's creation (*ibid*). Some of Lie's early ideas were developed in close collaboration with Felix Klein. Lie met with Klein every day from October 1869 through 1872: in Berlin from the end of October 1869 to the end of February 1870, and in Paris, Göttingen and Erlangen in the subsequent two years (*ibid*, p. 2). Lie stated that all of the principal results were obtained by 1884. But during the 1870s all his papers (except the very first note) were published in Norwegian journals, which impeded recognition of the work throughout the rest of Europe (*ibid*, p. 76). In 1884 a young German mathematician, Friedrich Engel, came to work with Lie on a systematic treatise to expose his theory of continuous groups. From this effort resulted the three-volume *Theorie der Transformationsgruppen*, published in 1888, 1890, and 1893.

Lie's ideas did not stand in isolation from the rest of mathematics. In fact, his interest in the geometry of differential equations was first motivated by the work of Carl Gustav Jacobi, on the theory of partial differential equations of first order and on the equations of classical mechanics. Much of Jacobi's work was published posthumously in the 1860s, generating enormous interest in France and Germany (Hawkins, p. 43). Lie's *idée fixe* was to develop a theory of symmetries of differential equations that would accomplish for them what Évariste Galois had done for algebraic equations: namely, to classify them in terms of group theory. Lie and other mathematicians showed that the most important equations for special functions and orthogonal polynomials tend to arise from group theoretical symmetries. Additional impetus to consider continuous groups came from ideas of Bernhard Riemann, on the foundations of geometry, and their further development in the hands of Klein. Thus three major themes in 19th century mathematics were combined by Lie in creating his new theory: the idea of symmetry, as exemplified by Galois through the algebraic notion of a group; geometric theory and the explicit solutions of differential equations of mechanics, worked out by Poisson and Jacobi; and the new understanding of geometry that emerged in the works of Plücker, Möbius, Grassmann and others, and culminated in Riemann's revolutionary vision of the subject.

Although today Sophus Lie is rightfully recognized as the creator of the theory of continuous groups, a major stride in the development of their structure theory, which was to have a profound influence on subsequent development of mathematics, was made by Wilhelm Killing, who in 1888 published the first paper in a series entitled *Die Zusammensetzung der stetigen endlichen Transformationsgruppen* (*The composition of continuous finite transformation groups*) (Hawkins, p. 100). The work of Killing, later refined and generalized by Élie Cartan, led to classification of semisimple Lie algebras, Cartan's theory of symmetric spaces, and Hermann Weyl's description of representations of compact and semisimple Lie groups using highest weights.

Weyl brought the early period of the development of the theory of Lie groups to fruition, for not only did he classify irreducible representations of semisimple Lie groups and connect the theory of groups with quantum mechanics, but he also put Lie's theory itself on firmer footing by clearly enunciating the distinction between Lie's *infinitesimal groups* (i.e., Lie algebras) and the Lie groups proper, and began investigations of topology of Lie groups (Borel (2001),). The theory of Lie groups was systematically reworked in modern mathematical language in a monograph by Claude Chevalley.

The concept of a Lie group, and possibilities of classification

Lie groups may be thought of as smoothly varying families of symmetries. Examples of symmetries include rotation about an axis. What must be understood is the nature of 'small' transformations, e.g., rotations through tiny angles, that link nearby transformations. The mathematical object capturing this structure is called a Lie algebra (Lie himself called them "infinitesimal groups"). It can be defined because Lie groups are manifolds, so have tangent spaces at each point.

The Lie algebra of any compact Lie group (very roughly: one for which the symmetries form a bounded set) can be decomposed as a direct sum of an abelian Lie algebra and some number of simple ones. The structure of an abelian Lie algebra is mathematically uninteresting (since the Lie bracket is identically zero); the interest is in the simple summands. Hence the question arises: what are the simple Lie algebras of compact groups? It turns out that they mostly fall into four infinite families, the "classical Lie algebras" A_n, B_n, C_n and D_n, which have simple descriptions in terms of symmetries of Euclidean space. But there are also just five "exceptional Lie algebras" that do not fall into any of these families. E_8 is the largest of these.

Properties

- The diffeomorphism group of a Lie group acts transitively on the Lie group
- Every Lie group is parallelizable, and hence an orientable manifold (there is a bundle isomorphism between its tangent bundle and the product of itself with the tangent space at the identity)

Types of Lie groups and structure theory

Lie groups are classified according to their algebraic properties (simple, semisimple, solvable, nilpotent, abelian), their connectedness (connected or simply connected) and their compactness.

- Compact Lie groups are all known: they are finite central quotients of a product of copies of the circle group S^1 and simple compact Lie groups (which correspond to connected Dynkin diagrams).
- Any simply connected solvable Lie group is isomorphic to a closed subgroup of the group of invertible upper triangular matrices of some rank, and any finite dimensional irreducible representation of such a group is 1 dimensional. Solvable groups are too messy to classify except in a few small dimensions.
- Any simply connected nilpotent Lie group is isomorphic to a closed subgroup of the group of invertible upper triangular matrices with 1's on the diagonal of some rank, and any finite dimensional irreducible representation of such a group is 1 dimensional. Like solvable groups, nilpotent groups are too messy to classify except in a few small dimensions.

- Simple Lie groups are sometimes defined to be those that are simple as abstract groups, and sometimes defined to be connected Lie groups with a simple Lie algebra. For example, $SL_2(\mathbf{R})$ is simple according to the second definition but not according to the first. They have all been classified (for either definition).
- Semisimple Lie groups are Lie groups whose Lie algebra is a product of simple Lie algebras.[1] They are central extensions of products of simple Lie groups.

The identity component of any Lie group is an open normal subgroup, and the quotient group is a discrete group. The universal cover of any connected Lie group is a simply connected Lie group, and conversely any connected Lie group is a quotient of a simply connected Lie group by a discrete normal subgroup of the center. Any Lie group G can be decomposed into discrete, simple, and abelian groups in a canonical way as follows. Write

G_{con} for the connected component of the identity

G_{sol} for the largest connected normal solvable subgroup

G_{nil} for the largest connected normal nilpotent subgroup

so that we have a sequence of normal subgroups

$$1 \subseteq G_{\mathrm{nil}} \subseteq G_{\mathrm{sol}} \subseteq G_{\mathrm{con}} \subseteq G.$$

Then

G/G_{con} is discrete

$G_{\mathrm{con}}/G_{\mathrm{sol}}$ is a central extension of a product of simple connected Lie groups.

$G_{\mathrm{sol}}/G_{\mathrm{nil}}$ is abelian. A connected abelian Lie group is isomorphic to a product of copies of $\mathbf{R}$ and the circle group S^1.

$G_{\mathrm{nil}}/1$ is nilpotent, and therefore its ascending central series has all quotients abelian.

This can be used to reduce some problems about Lie groups (such as finding their unitary representations) to the same problems for connected simple groups and nilpotent and solvable subgroups of smaller dimension.

The Lie algebra associated with a Lie group

To every Lie group, we can associate a Lie algebra, whose underlying vector space is the tangent space of G at the identity element, which completely captures the local structure of the group. Informally we can think of elements of the Lie algebra as elements of the group that are "infinitesimally close" to the identity, and the Lie bracket is something to do with the commutator of two such infinitesimal elements. Before giving the abstract definition we give a few examples:

- The Lie algebra of the vector space $\mathbf{R}^n$ is just $\mathbf{R}^n$ with the Lie bracket given by

$$[A, B] = 0.$$

(In general the Lie bracket of a connected Lie group is always 0 if and only if the Lie group is abelian.)

- The Lie algebra of the general linear group $GL_n(\mathbf{R})$ of invertible matrices is the vector space $M_n(\mathbf{R})$ of square matrices with the Lie bracket given by

$$[A, B] = AB - BA.$$

If G is a closed subgroup of $GL_n(\mathbf{R})$ then the Lie algebra of G can be thought of informally as the matrices m of $M_n(\mathbf{R})$ such that $1 + \varepsilon m$ is in G, where ε is an infinitesimal positive number with $\varepsilon^2 = 0$ (of course, no such real number ε exists). For example, the orthogonal group $O_n(\mathbf{R})$ consists of matrices A with $AA^{\mathrm{T}} = 1$, so the Lie algebra consists of the matrices m with $(1 + \varepsilon m)(1 + \varepsilon m)^{\mathrm{T}} = 1$, which is equivalent to $m + m^{\mathrm{T}} = 0$ because $\varepsilon^2 = 0$.

- Formally, when working over the reals, as here, this is accomplished by considering the limit as $\varepsilon \to 0$; but the "infinitesimal" language generalizes directly to Lie groups over general rings.

The concrete definition given above is easy to work with, but has some minor problems: to use it we first need to represent a Lie group as a group of matrices, but not all Lie groups can be represented in this way, and it is not obvious that the Lie algebra is independent of the representation we use. To get round these problems we give the general definition of the Lie algebra of any Lie group (in 4 steps):

1. Vector fields on any smooth manifold M can be thought of as derivations X of the ring of smooth functions on the manifold, and therefore form a Lie algebra under the Lie bracket $[X, Y] = XY - YX$, because the Lie bracket of any two derivations is a derivation.
2. If G is any group acting smoothly on the manifold M, then it acts on the vector fields, and the vector space of vector fields fixed by the group is closed under the Lie bracket and therefore also forms a Lie algebra.
3. We apply this construction to the case when the manifold M is the underlying space of a Lie group G, with G acting on $G = M$ by left translations $L_g(h) = gh$. This shows that the space of left invariant vector fields (vector fields satisfying $L_{g*}X_h = X_{gh}$ for every h in G, where L_{g*} denotes the differential of L_g) on a Lie group is a Lie algebra under the Lie bracket of vector fields.
4. Any tangent vector at the identity of a Lie group can be extended to a left invariant vector field by left translating the tangent vector to other points of the manifold. Specifically, the left invariant extension of an element v of the tangent space at the identity is the vector field defined by $v\hat{}_g = L_{g*}v$. This identifies the tangent space T_e at the identity with the space of left invariant vector fields, and therefore makes the tangent space at the identity into a Lie algebra, called the Lie algebra of G, usually denoted by a Fraktur $\mathfrak{g}$. Thus the Lie bracket on $\mathfrak{g}$ is given explicitly by $[v, w] = [v\hat{}, w\hat{}]_e$.

This Lie algebra $\mathfrak{g}$ is finite-dimensional and it has the same dimension as the manifold G. The Lie algebra of G determines G up to "local isomorphism", where two Lie groups are called **locally isomorphic** if they look the same near the identity element. Problems about Lie groups are often solved by first solving the corresponding problem for the Lie algebras, and the result for groups then usually follows easily. For example, simple Lie groups are usually classified by first classifying the corresponding Lie algebras.

We could also define a Lie algebra structure on T_e using right invariant vector fields instead of left invariant vector fields. This leads to the same Lie algebra, because the inverse map on G can be used to identify left invariant vector fields with right invariant vector fields, and acts as −1 on the tangent space T_e.

The Lie algebra structure on T_e can also be described as follows: the commutator operation

$$(x, y) \to xyx^{-1}y^{-1}$$

on $G \times G$ sends (e, e) to e, so its derivative yields a bilinear operation on T_eG. This bilinear operation is actually the zero map, but the second derivative, under the proper identification of tangent spaces, yields an operation that satisfies the axioms of a Lie bracket, and it is equal to twice the one defined through left-invariant vector fields.

Homomorphisms and isomorphisms

If G and H are Lie groups, then a Lie-group homomorphism $f : G \to H$ is a smooth group homomorphism. (It is equivalent to require only that f be continuous rather than smooth.) The composition of two such homomorphisms is again a homomorphism, and the class of all Lie groups, together with these morphisms, forms a category. Two Lie groups are called *isomorphic* if there exists a bijective homomorphism between them whose inverse is also a homomorphism. Isomorphic Lie groups are essentially the same; they only differ in the notation for their elements.

Every homomorphism $f : G \to H$ of Lie groups induces a homomorphism between the corresponding Lie algebras $\mathfrak{g}$ and $\mathfrak{h}$. The association $G \mapsto \mathfrak{g}$ is a functor (mapping between categories satisfying certain axioms).

One version of Ado's theorem is that every finite dimensional Lie algebra is isomorphic to a matrix Lie algebra. For every finite dimensional matrix Lie algebra, there is a linear group (matrix Lie group) with this algebra as its Lie algebra. So every abstract Lie algebra is the Lie algebra of some (linear) Lie group.

The *global structure* of a Lie group is not determined by its Lie algebra; for example, if Z is any discrete subgroup of the center of G then G and G/Z have the same Lie algebra (see the table of Lie groups for examples). A *connected* Lie group is simple, semisimple, solvable, nilpotent, or abelian if and only if its Lie algebra has the corresponding property.

If we require that the Lie group be simply connected, then the global structure is determined by its Lie algebra: for every finite dimensional Lie algebra $\mathfrak{g}$ over **F** there is a simply connected Lie group *G* with $\mathfrak{g}$ as Lie algebra, unique up to isomorphism. Moreover every homomorphism between Lie algebras lifts to a unique homomorphism between the corresponding simply connected Lie groups.

The exponential map

The exponential map from the Lie algebra $M_n(\mathbf{R})$ of the general linear group $GL_n(\mathbf{R})$ to $GL_n(\mathbf{R})$ is defined by the usual power series:

$$\exp(A) = 1 + A + \frac{A^2}{2!} + \frac{A^3}{3!} + \cdots$$

for matrices *A*. If *G* is any subgroup of $GL_n(\mathbf{R})$, then the exponential map takes the Lie algebra of *G* into *G*, so we have an exponential map for all matrix groups.

The definition above is easy to use, but it is not defined for Lie groups that are not matrix groups, and it is not clear that the exponential map of a Lie group does not depend on its representation as a matrix group. We can solve both problems using a more abstract definition of the exponential map that works for all Lie groups, as follows.

Every vector *v* in $\mathfrak{g}$ determines a linear map from **R** to $\mathfrak{g}$ taking 1 to *v*, which can be thought of as a Lie algebra homomorphism. Because **R** is the Lie algebra of the simply connected Lie group **R**, this induces a Lie group homomorphism $c : \mathbf{R} \to G$ so that

$$c(s+t) = c(s)c(t)$$

for all *s* and *t*. The operation on the right hand side is the group multiplication in *G*. The formal similarity of this formula with the one valid for the exponential function justifies the definition

$$\exp(v) = c(1).$$

This is called the *exponential map*, and it maps the Lie algebra $\mathfrak{g}$ into the Lie group *G*. It provides a diffeomorphism between a neighborhood of 0 in $\mathfrak{g}$ and a neighborhood of *e* in *G*. This exponential map is a generalization of the exponential function for real numbers (because **R** is the Lie algebra of the Lie group of positive real numbers with multiplication), for complex numbers (because **C** is the Lie algebra of the Lie group of non-zero complex numbers with multiplication) and for matrices (because $M_n(\mathbf{R})$ with the regular commutator is the Lie algebra of the Lie group $GL_n(\mathbf{R})$ of all invertible matrices).

Because the exponential map is surjective on some neighbourhood *N* of *e*, it is common to call elements of the Lie algebra **infinitesimal generators** of the group *G*. The subgroup of *G* generated by *N* is the identity component of *G*.

The exponential map and the Lie algebra determine the *local group structure* of every connected Lie group, because of the Baker–Campbell–Hausdorff formula: there exists a neighborhood *U* of the zero element of $\mathfrak{g}$, such that for *u*, *v* in *U* we have

$$\exp(u)\exp(v) = \exp(u + v + 1/2\,[u, v] + 1/12\,[[u, v], v] - 1/12\,[[u, v], u] - \ldots)$$

where the omitted terms are known and involve Lie brackets of four or more elements. In case *u* and *v* commute, this formula reduces to the familiar exponential law $\exp(u)\exp(v) = \exp(u + v)$.

The exponential map from the Lie algebra to the Lie group is not always onto, even if the group is connected (though it does map onto the Lie group for connected groups that are either compact or nilpotent). For example, the exponential map of $SL_2(\mathbf{R})$ is not surjective.

Infinite dimensional Lie groups

Lie groups are often defined to be finite dimensional, but there are many groups that resemble Lie groups, except for being infinite dimensional. The simplest was to define infinite dimensional Lie groups is to model them on Banach spaces, and in this case much of the basic theory is similar to that of finite dimensional lie groups. However this is inadequate for many applications, because many natural examples of infinite dimensional Lie groups are not Banach manifolds. Instead one needs to define Lie groups modeled on more general locally convex topological vector spaces. In this case the relation between the Lie algebra and the Lie group becomes rather subtle, and several results about finite dimensional Lie groups no longer hold.

Some of the examples that have been studied include:

- The group of diffeomorphisms of a manifold. Quite a lot is known about the group of diffeomorphisms of the circle. Its Lie algebra is (more or less) the Witt algebra, which has a central extension called the Virasoro algebra, used in string theory and conformal field theory. Very little is known about the diffeomorphism groups of manifolds of larger dimension. The diffeomorphism group of spacetime sometimes appears in attempts to quantize gravity.
- The group of smooth maps from a manifold to a finite dimensional Lie group is an example of a gauge group (with operation of pointwise multiplication), and is used in quantum field theory and Donaldson theory. If the manifold is a circle these are called loop groups, and have central extensions whose Lie algebras are (more or less) Kac–Moody algebras.
- There are infinite dimensional analogues of general linear groups, orthogonal groups, and so on. One important aspect is that these may have *simpler* topological properties: see for example Kuiper's theorem.

Notes

[1] Sigurdur Helgason, "Differential Geometry, Lie Groups, and Symmetric Spaces", Academic Press, 1978, page 131.

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Affine Lie algebra

In mathematics, an **affine Lie algebra** is an infinite-dimensional Lie algebra that is constructed in a canonical fashion out of a finite-dimensional simple Lie algebra. It is a Kac–Moody algebra whose generalized Cartan matrix is positive semi-definite and has corank 1. From purely mathematical point of view, affine Lie algebras are interesting because their representation theory, like representation theory of finite dimensional, semisimple Lie algebras is much better understood than that of general Kac–Moody algebras. As observed by Victor Kac, the character formula for representations of affine Lie algebras implies certain combinatorial identities, the **Macdonald identities**.

Affine Lie algebras play an important role in string theory and conformal field theory due to the way they are constructed: starting from a simple Lie algebra $\mathfrak{g}$, one considers the **loop algebra**, $L\mathfrak{g}$, formed by the $\mathfrak{g}$-valued functions on a circle (interpreted as the closed string) with pointwise commutator. The affine Lie algebra $\hat{\mathfrak{g}}$ is obtained by adding one extra dimension to the loop algebra and modifying a commutator in a non-trivial way, which physicists call a **quantum anomaly**. The point of view of string theory helps to understand many deep properties of affine Lie algebras, such as the fact that the characters of their representations are given by modular forms.

Affine Lie algebras from simple Lie algebras

Construction

If $\mathfrak{g}$ is a finite dimensional simple Lie algebra, the corresponding affine Lie algebra $\hat{\mathfrak{g}}$ is constructed as a central extension of the infinite-dimensional Lie algebra $\mathfrak{g}\otimes\mathbb{C}[t,t^{-1}]$, with one-dimensional center $\mathbb{C}c.$ As a vector space,

$$\hat{\mathfrak{g}}=\mathfrak{g}\otimes\mathbb{C}[t,t^{-1}]\oplus\mathbb{C}c,$$

where $\mathbb{C}[t,t^{-1}]$ is the complex vector space of Laurent polynomials in the indeterminate *t*. The Lie bracket is defined by the formula

$$[a\otimes t^n+\alpha c,b\otimes t^m+\beta c]=[a,b]\otimes t^{n+m}+\langle a|b\rangle n\delta_{m+n,0}c$$

for all $a,b\in\mathfrak{g},\alpha,\beta\in\mathbb{C}$ and $n,m\in\mathbb{Z}$, where $[a,b]$ is the Lie bracket in the Lie algebra $\mathfrak{g}$ and $\langle\cdot|\cdot\rangle$ is the Cartan-Killing form on $\mathfrak{g}$.

The affine Lie algebra corresponding to a finite-dimensional semisimple Lie algebra is the direct sum of the affine Lie algebras corresponding to its simple summands.

Constructing the Dynkin diagrams

The Dynkin diagram of each affine Lie algebra consists of that of the corresponding simple Lie algebra plus an additional node, which corresponds to the addition of an imaginary root. Of course, such a node cannot be attached to the Dynkin diagram in just any location, but for each simple Lie algebra there exists a number of possible attachments equal to the cardinality of the group of outer automorphisms of the Lie algebra. In particular, this group always contains the identity element, and the corresponding affine Lie algebra is called an **untwisted** affine Lie algebra. When the simple algebra admits automorphisms that are not inner automorphisms, one may obtain other

Dynkin diagrams and these correspond to twisted affine Lie algebras.

Classifying the central extensions

The attachment of an extra node to the Dynkin diagram of the corresponding simple Lie algebra corresponds to the following construction. An affine Lie algebra can always be constructed as a central extension of the loop algebra of the corresponding simple Lie algebra. If one wishes to begin instead with a semisimple Lie algebra, then one needs to centrally extend by a number of elements equal to the number of simple components of the semisimple algebra. In physics, one often considers instead the direct sum of a semisimple algebra and an abelian algebra $\mathbb{C}^n$. In this case one also needs to add *n* further central elements for the *n* abelian generators.

The second integral cohomology of the loop group of the corresponding simple compact Lie group is isomorphic to the integers. Central extensions of the affine Lie group by a single generator are topologically circle bundles over this free loop group, which are classified by a two-class known as the first Chern class of the fibration. Therefore the central extensions of an affine Lie group are classified by a single parameter k which is called the central charge in the physics literature, where it first appeared. Unitary highest weight representations of the affine compact groups only exist when k is a natural number. More generally, if one considers a semi-simple algebra, there is a central charge for each simple component.

Applications

They appear naturally in theoretical physics (for example, in conformal field theories such as the WZW model and coset models and even on the worldsheet of the heterotic string), geometry, and elsewhere in mathematics.

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Kac–Moody algebra

In mathematics, a **Kac–Moody algebra** (named for Victor Kac and Robert Moody, who independently discovered them) is a Lie algebra, usually infinite-dimensional, that can be defined by generators and relations through a generalized Cartan matrix. These algebras form a generalization of finite-dimensional semisimple Lie algebras, and many properties related to the structure of a Lie algebra such as its root system, irreducible representations, and connection to flag manifolds have natural analogues in the Kac–Moody setting.

A class of **Kac–Moody algebras** called **affine Lie algebras** is of particular importance in mathematics and theoretical physics, especially conformal field theory and the theory of exactly solvable models. Kac discovered an elegant proof of certain combinatorial identities, Macdonald identities, which is based on the representation theory of affine Kac–Moody algebras. H. Garland and Jim Lepowsky demonstrated that Rogers-Ramanujan identities can be derived in a similar fashion.[1]

History of Kac-Moody algebras

The initial construction by Élie Cartan and Wilhelm_Killing of finite dimensional simple Lie algebras from the Cartan integers was type dependent. In 1966 Jean-Pierre Serre showed that relations of Claude Chevalley and Harish-Chandra[2] , with simplifications by N. Jacobson[3] , give a defining presentation for the Lie algebra[4] . One could thus describe a simple Lie algebra in terms of generators and relations using data from the matrix of Cartan integers, which is naturally positive definite.

In his 1967 thesis, Robert Moody considered Lie algebras whose Cartan matrix is no longer positive definite[5] [6] . This still gave rise to a Lie algebra, but one which is now infinite dimensional. Simultaneously, **Z**-graded Lie algebras were being studied in Moscow where I. L. Kantor introduced and studied a general class of Lie algebras including what eventually became known as **Kac-Moody algebras**[7] . Victor Kac was also studying simple or nearly simple Lie algebras with polynomial growth. A rich mathematical theory of infinite dimensional Lie algebras evolved. An account of the subject, which also includes works of many others is given in (Kac 1990).[8] See also (Seligman 1987)[9] .

Definition

A Kac–Moody algebra is given by the following:

1. An $n \times n$ generalized Cartan matrix $C = (c_{ij})$ of rank r.
2. A vector space $\mathfrak{h}$ over the complex numbers of dimension $2n - r$.
3. A set of n linearly independent elements α_i of $\mathfrak{h}$ and a set of n linearly independent elements α_i^* of the dual space, such that $\alpha_i^*(\alpha_j) = c_{ij}$. The α_i are known as **coroots**, while the α_i^* are known as **roots**.

The Kac–Moody algebra is the Lie algebra $\mathfrak{g}$ defined by generators e_i and f_i and the elements of $\mathfrak{h}$ and relations

- $[e_i, f_i] = \alpha_i$
- $[e_i, f_j] = 0$ for $i \neq j$
- $[e_i, x] = \alpha_i^*(x) e_i$, for $x \in \mathfrak{h}$
- $[f_i, x] = -\alpha_i^*(x) f_i$, for $x \in \mathfrak{h}$
- $[x, x'] = 0$ for $x, x' \in \mathfrak{h}$
- $\mathrm{ad}(e_i)^{1-c_{ij}}(e_j) = 0$
- $\mathrm{ad}(f_i)^{1-c_{ij}}(f_j) = 0$

where $\mathrm{ad} : \mathfrak{g} \to \mathrm{End}(\mathfrak{g}), \mathrm{ad}(x)(y) = [x, y]$ is the adjoint representation of $\mathfrak{g}$.

A real (possibly infinite-dimensional) Lie algebra is also considered a Kac–Moody algebra if its complexification is a Kac–Moody algebra.

Interpretation

$\mathfrak{h}$ is a Cartan subalgebra of the Kac–Moody algebra.

If g is an element of the Kac–Moody algebra such that

$$\forall x \in \mathfrak{h},\ [g,x] = \omega(x)g$$

where ω is an element of $\mathfrak{h}^*$, then g is said to have weight ω. The Kac–Moody algebra can be diagonalized into weight eigenvectors. The Cartan subalgebra h has weight zero, e_i has weight α^*_i and f_i has weight $-\alpha^*_i$. If the Lie bracket of two weight eigenvectors is nonzero, then its weight is the sum of their weights. The condition $[e_i, f_j] = 0$ for $i \neq j$ simply means the α^*_i are simple roots.

Types of Kac–Moody algebras

Properties of a Kac–Moody algebra are controlled by the algebraic properties of its generalized Cartan matrix C. In order to classify Kac–Moody algebras, it is enough to consider the case of an *indecomposable* matrix C, that is, assume that there is no decomposition of the set of indices I into a disjoint union of non-empty subsets I_1 and I_2 such that $C_{ij} = 0$ for all i in I_1 and j in I_2. Any decomposition of the generalized Cartan matrix leads to the direct sum decomposition of the corresponding Kac–Moody algebra:

$$\mathfrak{g}(C) \simeq \mathfrak{g}(C_1) \oplus \mathfrak{g}(C_2),$$

where the two Kac–Moody algebras in the right hand side are associated with the submatrices of C corresponding to the index sets I_1 and I_2.

An important subclass of Kac–Moody algebras corresponds to *symmetrizable* generalized Cartan matrices C, which can be decomposed as DS, where D is a diagonal matrix with positive integer entries and S is a symmetric matrix. Under the assumptions that C is symmetrizable and indecomposable, the Kac–Moody algebras are divided into three classes:

- A positive definite matrix S gives rise to a finite-dimensional simple Lie algebra.
- A positive semidefinite matrix S gives rise to an infinite-dimensional Kac–Moody algebra of **affine type**, or an affine Lie algebra.
- An indefinite matrix S gives rise to a Kac–Moody algebra of **indefinite type**.
- Since the diagonal entries of C and S are positive, S cannot be negative definite or negative semidefinite.

Symmetrizable indecomposable generalized Cartan matrices of finite and affine type have been completely classified. They correspond to Dynkin diagrams and affine Dynkin diagrams. Very little is known about the Kac–Moody algebras of indefinite type. Among those, the main focus has been on the (generalized) Kac–Moody algebras of **hyperbolic type**, for which the matrix S is indefinite, but for each proper subset of I, the corresponding submatrix is positive definite or positive semidefinite. Such matrices have rank at most 10 and have also been completely determined.[10]

Notes

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- SIGMA: Special Issue on Kac-Moody Algebras and Applications (http://www.emis.de/journals/SIGMA/Kac-Moody_algebras.html)

Hopf algebra

In mathematics, a **Hopf algebra**, named after Heinz Hopf, is a structure that is simultaneously a (unital associative) algebra, a (counital coassociative) coalgebra, with these structures compatible making it a bialgebra, and moreover is equipped with an antiautomorphism satisfying a certain property.

Hopf algebras occur naturally in algebraic topology, where they originated and are related to the H-space concept, in group scheme theory, in group theory (via the concept of a group ring), and in numerous other places, making them probably the most familiar type of bialgebra. Hopf algebras are also studied in their own right, with much work on specific classes of examples on the one hand and classification problems on the other.

Formal definition

Formally, a Hopf algebra is a bialgebra *H* over a field *K* together with a *K*-linear map $S \colon H \to H$ (called the **antipode**) such that the following diagram commutes:

$$\begin{tikzcd}
& H\otimes H \arrow[rr, "S\otimes\mathrm{id}"] & & H\otimes H \arrow[dr, "\nabla"] & \\
H \arrow[ur, "\Delta"] \arrow[rr, "\varepsilon"] \arrow[dr, "\Delta"'] & & K \arrow[rr, "\eta"] & & H \\
& H\otimes H \arrow[rr, "\mathrm{id}\otimes S"'] & & H\otimes H \arrow[ur, "\nabla"'] &
\end{tikzcd}$$

Here Δ is the comultiplication of the bialgebra, ∇ its multiplication, η its unit and ε its counit. In the sumless Sweedler notation, this property can also be expressed as

$$S(c_{(1)})c_{(2)} = c_{(1)}S(c_{(2)}) = \epsilon(c)1 \qquad \text{for all } c \in H.$$

As for algebras, one can replace the underlying field *K* with a commutative ring *R* in the above definition.

The definition of Hopf algebra is self-dual (as reflected in the symmetry of the above diagram), so if one can define a dual of *H* (which is always possible if *H* is finite-dimensional), then it is automatically a Hopf algebra.

Properties of the antipode

The antipode *S* is sometimes required to have a *K*-linear inverse, which is automatic in the finite-dimensional case, or if *H* is commutative or cocommutative (or more generally quasitriangular).

In general, *S* is an antihomomorphism,[1] so S^2 is a homomorphism, which is therefore an automorphism if *S* was invertible (as may be required).

If $S^2 = Id$, then the Hopf algebra is said to be **involutive** (and the underlying algebra with involution is a *-algebra). If *H* is finite-dimensional semisimple over a field of characteristic zero, commutative, or cocommutative, then it is involutive.

If a bialgebra *B* admits an antipode *S*, then *S* is unique ("a bialgebra admits at most 1 Hopf algebra structure").[2]

The antipode is an analog to the inversion map on a group that sends g to g^{-1}.[3]

Hopf subalgebras

A subalgebra K (not to be confused with the Field K in the notation above) of a Hopf algebra H is a Hopf subalgebra if it is a subcoalgebra of H and the antipode S maps K into K. In other words, a Hopf subalgebra K is a Hopf algebra in its own right when the multiplication, comultiplication, counit and antipode of H is restricted to K (and additionally the identity 1 is required to be in K). The Nichols-Zoeller Freeness theorem established (in 1989) that either natural K-module H is free of finite rank if H is finite dimensional: a generalization of Lagrange's theorem for subgroups. As a corollary of this and integral theory, a Hopf subalgebra of a semisimple finite dimensional Hopf algebra is automatically semisimple.

A Hopf subalgebra K is said to be right normal in a Hopf algebra H if it satisfies the condition of stability, $ad_r(h)(K) \subseteq K$ for all h in H, where the right adjoint mapping ad_r is defined by $ad_r(h)(k) = S(h_{(1)})kh_{(2)}$ for all k in K, h in H. Similarly, a Hopf subalgebra K is left normal in H if it is stable under the left adjoint mapping defined by $ad_\ell(h)(k) = h_{(1)}kS(h_{(2)})$. The two conditions of normality are equivalent if the antipode S is bijective, in which case K is said to be a normal Hopf subalgebra.
A normal Hopf subalgebra K in H satisfies the condition (of equality of subsets of H): $HK^+ = K^+H$ where K^+ denotes the kernel of the counit on K. This normality condition implies that HK^+ is a Hopf ideal of H (i.e. an algebra ideal in the kernel of the counit, a coalgebra coideal and stable under the antipode). As a consequence one has a quotient Hopf algebra H/HK^+ and epimorphism $H \to H/K^+H$, a theory analogous to that of normal subgroups and quotient groups in group theory.[4]

Examples

Group algebra. Suppose *G* is a group. The group algebra *KG* is a unital associative algebra over *K*. It turns into a Hopf algebra if we define

- $\Delta : KG \to KG \otimes KG$ by $\Delta(g) = g \otimes g$ for all g in G
- $\varepsilon : KG \to K$ by $\varepsilon(g) = 1$ for all g in G
- $S : KG \to KG$ by $S(g) = g^{-1}$ for all g in G.

Functions on a finite group. Suppose now that *G* is a *finite* group. Then the set K^G of all functions from *G* to *K* with pointwise addition and multiplication is a unital associative algebra over *K*, and $K^G \otimes K^G$ is naturally isomorphic to $K^{G\times G}$ (for *G* infinite, $K^G \otimes K^G$ is a proper subset of $K^{G\times G}$). The set K^G becomes a Hopf algebra if we define

- $\Delta : K^G \to K^{G\times G}$ by $\Delta(f)(x,y) = f(xy)$ for all f in K^G and all x,y in G
- $\varepsilon : K^G \to K$ by $\varepsilon(f) = f(e)$ for every f in K^G [here e is the identity element of G]
- $S : K^G \to K^G$ by $S(f)(x) = f(x^{-1})$ for all f in K^G and all x in G.

Note that functions on a finite group can be identified with the group ring, though these are more naturally thought of as dual – the group ring consists of *finite* sums of elements, and thus pairs with functions on the group by evaluating the function on the summed elements.

Regular functions on an algebraic group. Generalizing the previous example, we can use the same formulas to show that for a given algebraic group *G* over *K*, the set of all regular functions on *G* forms a Hopf algebra.

Universal enveloping algebra. Suppose *g* is a Lie algebra over the field *K* and *U* is its universal enveloping algebra. *U* becomes a Hopf algebra if we define

- $\Delta : U \to U \otimes U$ by $\Delta(x) = x \otimes 1 + 1 \otimes x$ for every x in g (this rule is compatible with commutators and can therefore be uniquely extended to all of U).
- $\varepsilon : U \to K$ by $\varepsilon(x) = 0$ for all x in g (again, extended to U)
- $S : U \to U$ by $S(x) = -x$ for all x in g.

Cohomology of Lie groups

The cohomology algebra of a Lie group is a Hopf algebra: the multiplication is provided by the cup-product, and the comultiplication

$$H^*(G) \to H^*(G \times G) \cong H^*(G) \otimes H^*(G)$$

by the group multiplication $G \times G \to G$. This observation was actually a source of the notion of Hopf algebra. Using this structure, Hopf proved a structure theorem for the cohomology algebra of Lie groups.

Theorem (Hopf)[5] Let *A* be a finite-dimensional, graded commutative, graded cocommutative Hopf algebra over a field of characteristic 0. Then *A* (as an algebra) is a free exterior algebra with generators of odd degree.

Quantum groups and non-commutative geometry

All examples above are either commutative (i.e. the multiplication is commutative) or co-commutative (i.e. $\Delta = T \circ \Delta$ where T: $H \otimes H \to H \otimes H$ is defined by $T(x \otimes y) = y \otimes x$). Other interesting Hopf algebras are certain "deformations" or "quantizations" of those from example 3 which are neither commutative nor co-commutative. These Hopf algebras are often called *quantum groups*, a term that is so far only loosely defined. They are important in noncommutative geometry, the idea being the following: a standard algebraic group is well described by its standard Hopf algebra of regular functions; we can then think of the deformed version of this Hopf algebra as describing a certain "non-standard" or "quantized" algebraic group (which is not an algebraic group at all). While there does not seem to be a direct way to define or manipulate these non-standard objects, one can still work with their Hopf algebras, and indeed one *identifies* them with their Hopf algebras. Hence the name "quantum group".

Related concepts

Graded Hopf algebras are often used in algebraic topology: they are the natural algebraic structure on the direct sum of all homology or cohomology groups of an H-space.

Locally compact quantum groups generalize Hopf algebras and carry a topology. The algebra of all continuous functions on a Lie group is a locally compact quantum group.

Quasi-Hopf algebras are generalizations of Hopf algebras, where coassociativity only holds up to a twist.

Weak Hopf algebras, or quantum groupoids, are generalizations of Hopf algebras. Like Hopf algebras, weak Hopf algebras form a self-dual class of algebras; i.e., if H is a (weak) Hopf algebra, so is H^*, the dual space of linear forms on H (with respect to the algebra-coalgebra structure obtained from the natural pairing with H and its coalgebra-algebra structure).

A weak Hopf algebra H is usually taken to be a 1) finite dimensional algebra and coalgebra with coproduct $\Delta : H \to H \otimes H$ and counit $\epsilon : H \to k$ satisfying all the axioms of Hopf algebra except possibly $\Delta(1) \neq 1 \otimes 1$ or $\epsilon(ab) \neq \epsilon(a)\epsilon(b)$ for some a,b in H. Instead one requires that $(\Delta(1) \otimes 1)(1 \otimes \Delta(1)) = (\Delta(1) \otimes 1)(1 \otimes \Delta(1)) = (\Delta \otimes \mathrm{Id})\Delta(1)$ and $\epsilon(abc) = \sum \epsilon(ab_{(1)})\epsilon(b_{(2)}c) = \sum \epsilon(ab_{(2)})\epsilon(b_{(1)}c)$ for all a,b, and c in H.

2) H has a weakened antipode $S : H \to H$ satisfying the axioms (a) - (c): (a) $S(a_{(1)})a_{(2)} = 1_{(1)}\epsilon(a1_{(2)})$ for all a in H (the right-hand side is the interesting projection usually denoted by $\Pi^R(a)$ or $\epsilon_s(a)$ with image a separable subalgebra denoted by H^R or H_s); (b) $a_{(1)}S(a_{(2)} = \epsilon(1_{(1)}a)1_{(2)}$ for all a in H (another interesting projection usually denoted by $\Pi^L(a)$ or $\epsilon_t(a)$ with image a separable algebra H^L or H_t, anti-isomorphic to H^L via S); and (c) $S(a_{(1)})a_{(2)}S(a_{(3)} = S(a)$ for all a in H. Note that if $\Delta(1) = 1 \otimes 1$, these conditions reduce to the two usual conditions on the antipode of a Hopf algebra.

The axioms are partly chosen so that the category of H-modules is a rigid tensor category. The unit H-module is the separable algebra H^L mentioned above.

For example, a finite groupoid algebra is a weak Hopf algebra. In particular, the groupoid algebra on [n] with one pair of invertible arrows e_{ij} and e_{ji} between i and j in [n] is isomorphic to the algebra H of n x n matrices. The weak Hopf algebra structure on this particular H is given by coproduct $\Delta(e_{ij}) = e_{ij} \otimes e_{ij}$, counit $\epsilon(e_{ij}) = 1$ and antipode $S(e_{ij}) = e_{ji}$. The separable subalgebras H^L and H^R coincide and are non-central commutative algebras in this particular case (the subalgebra of diagonal matrices).
Early theoretical contributions to weak Hopf algebras are to be found in [6] as well as [7]

Hopf algebroids introduced by J.-H. Lu in 1996 as a result on work on groupoids in Poisson geometry (later shown equivalent in nontrivial way to a construction of Takeuchi from the 1970s and another by Xu around the year 2000): Hopf algebroids generalize weak Hopf algebras and certain skew Hopf algebras. They may be loosely thought of as Hopf algebras over a noncommutative base ring, where weak Hopf algebras become Hopf algebras over a separable algebra. It is a theorem that a Hopf algebroid satisfying a finite projectivity condition over a separable algebra is a weak Hopf algebra, and conversely a weak Hopf algebra H is a Hopf algebroid over its separable subalgebra H^L. The antipode axioms have been changed by G. Böhm and K. Szlachanyi (J. Algebra) in 2004 for tensor categorical reasons and to accommodate examples associated to depth two Frobenius algebra extensions.

A left Hopf algebroid (H,R) is a left bialgebroid together with an antipode: the bialgebroid (H,R) consists of a total algebra H and a base algebra R and two mappings, an algebra homomorphism $s : R \to H$ called a source map, an algebra anti-homomorphism $t : R \to H$ called a target map, such that the commutativity condition $s(r_1)t(r_2) = t(r_2)s(r_1)$ is satisfied for all $r_1, r_2 \in R$. The axioms resemble those of a Hopf algebra but are complicated by the possibility that R is a noncommutative algebra or its images under s and t are not in the center of H. In particular a left bialgebroid (H,R) has an R-R-bimodule structure on H which prefers the left side as follows: $r_1 \cdot h \cdot r_2 = s(r_1)t(r_2)h$ for all h in H, $r_1, r_2 \in R$. There is a coproduct $\Delta : H \to H \otimes_R H$ and counit $\epsilon : H \to R$ that make (H, R, Δ, ϵ) an R-coring (with axioms like that of a coalgebra such that all mappings are R-R-bimodule homomorphisms and all tensors over R). Additionally the bialgebroid (H,R) must satisfy $\Delta(ab) = \Delta(a)\Delta(b)$ for all a,b in H, and a condition to make sure this last condition makes sense: every image point $\Delta(a)$ satisfies $a_{(1)}t(r) \otimes a_{(2)} = a_{(1)} \otimes a_{(2)}s(r)$ for all r in R. Also $\Delta(1) = 1 \otimes 1$. The counit is required to satisfy $\epsilon(1_H) = 1_R$ and the condition $\epsilon(ab) = \epsilon(as(\epsilon(b))) = \epsilon(at(\epsilon(b)))$.
The antipode $S : H \to H$ is usually taken to be an algebra anti-automorphism satisfying conditions of exchanging the source and target maps and satisfying two axioms like Hopf algebra antipode axioms; see the references in Lu or in Böhm-Szlachanyi for a more example-category friendly, though somewhat more complicated, set of axioms for the antipode S. The latter set of axioms depend on the axioms of a right bialgebroid as well, which are a straightforward switching of left to right, s with t, of the axioms for a left bialgebroid given above.

As an example of left bialgebroid, take R to be any algebra over a field k. Let H be its algebra of linear self-mappings. Let s(r) be left multication by r on R; let t(r) be right multiplication by r on R. H is a left bialgebroid over R, which may be seen as follows. From the fact that $H \otimes_R H \cong \mathrm{Hom}_k(R \otimes R, R)$ one may define a coproduct by $\Delta(f)(r \otimes u) = f(ru)$ for each linear transformation f from R to itself and all r,u in R. Coassociativity of the coproduct follows from associativity of the product on R. A counit is given by $\epsilon(f) = f(1)$. The counit axioms of a coring follow from the identity element condition on multiplication in R. The reader will be amused, or at least edified, to check that (H,R) is a left bialgebroid. In case R is an Azumaya algebra, in which case H is isomorphic to R tensor R, an antipode comes from transposing tensors, which makes H a Hopf algebroid over R.
Multiplier Hopf algebras introduced by Alfons Van Daele in 1994[8] are generalizations of Hopf algebras where comultiplication from an algebra (with or withthout unit) to the multiplier algebra of tensor product algebra of the algebra with itself.

Hopf group-(co)algebras introduced by V.G.Turaev in 2000 are also generalizations of Hopf algebras.

Analogy with groups

Groups can be axiomatized by the same diagrams (equivalently, operations) as a Hopf algebra, where *G* is taken to be a set instead of a module. In this case:

- the field *K* is replaced by the 1-point set
- there is a natural counit (map to 1 point)
- there is a natural comultiplication (the diagonal map)
- the unit is the identity element of the group
- the multiplication is the multiplication in the group
- the antipode is the inverse

In this philosophy, a group can be thought of as a Hopf algebra over the "field with one element".[9]

See also

- Quasitriangular Hopf algebra
- Algebra/set analogy
- Representation theory of Hopf algebras
- Ribbon Hopf algebra
- Superalgebra
- Supergroup
- Anyonic Lie algebra

Notes

[1] Dăscălescu, Năstăsescu & Raianu (2001), Prop. 4.2.6, p. 153 (http://books.google.com/books?id=pBJ6sbPHA0IC&pg=PA153&dq="is+an+antimorphism+of+algebras")
[2] Dăscălescu, Năstăsescu & Raianu (2001), Remarks 4.2.3, p. 151 (http://books.google.com/books?id=pBJ6sbPHA0IC&pg=PA151&dq="the+antipode+is+unique")
[3] Quantum groups lecture notes (http://www.mathematik.uni-muenchen.de/~pareigis/Vorlesungen/98SS/Quantum_Groups/LN2_1.PDF)
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[5] Hopf, 1941.
[6] Gabriella Böhm, Florian Nill, Kornel Szlachanyi. J. Algebra 221 (1999), 385-438
[7] Dmitri Nikshych, Leonid Vainerman, in: New direction in Hopf algebras, S. Montgomery and H.-J. Schneider, eds., M.S.R.I. Publications, vol. 43, Cambridge, 2002, 211-262.
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[9] Group = Hopf algebra « Secret Blogging Seminar (http://sbseminar.wordpress.com/2007/10/07/group-hopf-algebra/), Group objects and Hopf algebras (http://www.youtube.com/watch?v=p3kkm5dYH-w), video of Simon Willerton.

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Quantum group

In mathematics and theoretical physics, the term **quantum group** denotes various kinds of noncommutative algebra with additional structure. In general, a quantum group is some kind of Hopf algebra. There is no single, all-encompassing definition, but instead a family of broadly similar objects.

The term "quantum group" often denotes a kind of noncommutative algebra with additional structure that first appeared in the theory of quantum integrable systems, and which was then formalized by Vladimir Drinfel'd and Michio Jimbo as a particular class of Hopf algebra. The same term is also used for other Hopf algebras that deform or are close to classical Lie groups or Lie algebras, such as a `bicrossproduct' class of quantum groups introduced by Shahn Majid a little after the work of Drinfeld and Jimbo.

In Drinfeld's approach, quantum groups arise as Hopf algebras depending on an auxiliary parameter *q* or *h*, which become universal enveloping algebras of a certain Lie algebra, frequently semisimple or affine, when $q = 1$ or $h = 0$. Closely related are certain dual objects, also Hopf algebras and also called quantum groups, deforming the algebra of functions on the corresponding semisimple algebraic group or a compact Lie group.

Just as groups often appear as symmetries, quantum groups act on many other mathematical objects and it has become fashionable to introduce the adjective *quantum* in such cases; for example there are quantum planes and quantum Grassmannians.

Intuitive meaning

The discovery of quantum groups was quite unexpected, since it was known for a long time that compact groups and semisimple Lie algebras are "rigid" objects, in other words, they cannot be "deformed". One of the ideas behind quantum groups is that if we consider a structure that is in a sense equivalent but larger, namely a group algebra or a universal enveloping algebra, then a group or enveloping algebra can be "deformed", although the deformation will no longer remain a group or enveloping algebra. More precisely, deformation can be accomplished within the category of Hopf algebras that are not required to be either commutative or cocommutative. One can think of the deformed object as an algebra of functions on a "noncommutative space", in the spirit of the noncommutative geometry of Alain Connes. This intuition, however, came after particular classes of quantum groups had already proved their usefulness in the study of the quantum Yang-Baxter equation and quantum inverse scattering method developed by the Leningrad School (Ludwig Faddeev, Leon Takhtajan, Evgenii Sklyanin, Nicolai Reshetikhin and Korepin) and related work by the Japanese School.[1] The intuition behind the second, bicrossproduct, class of quantum groups was different and came from the search for self-dual objects as an approach to quantum gravity[2] .

Drinfel'd-Jimbo type quantum groups

One type of objects commonly called a "quantum group" appeared in the work of Vladimir Drinfel'd and Michio Jimbo as a deformation of the universal enveloping algebra of a semisimple Lie algebra or, more generally, a Kac-Moody algebra, in the category of Hopf algebras. The resulting algebra has additional structure, making it into a quasitriangular Hopf algebra.

Let $A = (a_{ij})$ be the Cartan matrix of the Kac-Moody algebra, and let *q* be a nonzero complex number distinct from 1, then the quantum group, $U_q(G)$, where *G* is the Lie algebra whose Cartan matrix is *A*, is defined as the unital associative algebra with generators k_λ (where λ is an element of the weight lattice, *i.e.* $2(\lambda, \alpha_i)/(\alpha_i, \alpha_i) \in \mathbb{Z}$ for all *i*), and e_i and f_i (for simple roots, α_i), subject to the following relations:

- $k_0 = 1$,
- $k_\lambda k_\mu = k_{\lambda+\mu}$,
- $k_\lambda e_i k_\lambda^{-1} = q^{(\lambda,\alpha_i)} e_i$,

- $k_\lambda f_i k_\lambda^{-1} = q^{-(\lambda,\alpha_i)} f_i$,
- $[e_i, f_j] = \delta_{ij} \frac{k_i - k_i^{-1}}{q_i - q_i^{-1}}$,
- $\sum_{n=0}^{1-a_{ij}} (-1)^n \frac{[1-a_{ij}]_{q_i}!}{[1-a_{ij}-n]_{q_i}![n]_{q_i}!} e_i^n e_j e_i^{1-a_{ij}-n} = 0$, for $i \neq j$,
- $\sum_{n=0}^{1-a_{ij}} (-1)^n \frac{[1-a_{ij}]_{q_i}!}{[1-a_{ij}-n]_{q_i}![n]_{q_i}!} f_i^n f_j f_i^{1-a_{ij}-n} = 0$, for $i \neq j$,

where $k_i = k_{\alpha_i}$, $q_i = q^{\frac{1}{2}(\alpha_i,\alpha_i)}$, $[0]_{q_i}! = 1$, $[n]_{q_i}! = \prod_{m=1}^{n} [m]_{q_i}$ for all positive integers n, and $[m]_{q_i} = \frac{q_i^m - q_i^{-m}}{q_i - q_i^{-1}}$. These are the q-factorial and q-number, respectively, the q-analogs of the ordinary factorial.

The last two relations above are the *q*-Serre relations, the deformations of the Serre relations.

In the limit as $q \to 1$, these relations approach the relations for the universal enveloping algebra $U(G)$, where $k_\lambda \to 1$ and $\frac{k_\lambda - k_{-\lambda}}{q - q^{-1}} \to t_\lambda$ as $q \to 1$, where the element, t_λ, of the Cartan subalgebra satisfies $(t_\lambda, h) = \lambda(h)$ for all *h* in the Cartan subalgebra.

There are various coassociative coproducts under which these algebras are Hopf algebras, for example,

- $\Delta_1(k_\lambda) = k_\lambda \otimes k_\lambda$, $\Delta_1(e_i) = 1 \otimes e_i + e_i \otimes k_i$, $\Delta_1(f_i) = k_i^{-1} \otimes f_i + f_i \otimes 1$,
- $\Delta_2(k_\lambda) = k_\lambda \otimes k_\lambda$, $\Delta_2(e_i) = k_i^{-1} \otimes e_i + e_i \otimes 1$, $\Delta_2(f_i) = 1 \otimes f_i + f_i \otimes k_i$,
- $\Delta_3(k_\lambda) = k_\lambda \otimes k_\lambda$, $\Delta_3(e_i) = k_i^{-\frac{1}{2}} \otimes e_i + e_i \otimes k_i^{\frac{1}{2}}$, $\Delta_3(f_i) = k_i^{-\frac{1}{2}} \otimes f_i + f_i \otimes k_i^{\frac{1}{2}}$, where the set of generators has been extended, if required, to include k_λ for λ which is expressible as the sum of an element of the weight lattice and half an element of the root lattice.

In addition, any Hopf algebra leads to another with reversed copproduct $T \circ \Delta$, where T is given by $T(x \otimes y) = y \otimes x$, giving three more possible versions.

The counit on $U_q(A)$ is the same for all these coproducts: $\epsilon(k_\lambda) = 1$, $\epsilon(e_i) = 0$, $\epsilon(f_i) = 0$, and the respective antipodes for the above coproducts are given by

- $S_1(k_\lambda) = k_{-\lambda}$, $S_1(e_i) = -e_i k_i^{-1}$, $S_1(f_i) = -k_i f_i$,
- $S_2(k_\lambda) = k_{-\lambda}$, $S_2(e_i) = -k_i e_i$, $S_2(f_i) = -f_i k_i^{-1}$,
- $S_3(k_\lambda) = k_{-\lambda}$, $S_3(e_i) = -q_i e_i$, $S_3(f_i) = -q_i^{-1} f_i$.

Alternatively, the quantum group $U_q(G)$ can be regarded as an algebra over the field $\mathbb{C}(q)$, the field of all rational functions of an indeterminate *q* over $\mathbb{C}$.

Similarly, the quantum group $U_q(G)$ can be regarded as an algebra over the field $\mathbb{Q}(q)$, the field of all rational functions of an indeterminate *q* over $\mathbb{Q}$ (see below in the section on quantum groups at *q* = 0). The center of quantum group can be described by quantum determinant.

Representation theory

Just as there are many different types of representations for Kac-Moody algebras and their universal enveloping algebras, so there are many different types of representation for quantum groups.

As is the case for all Hopf algebras, $U_q(G)$ has an adjoint representation on itself as a module, with the action being given by $\mathrm{Ad}_x \cdot y = \sum_{(x)} x_{(1)} y S(x_{(2)})$, where $\Delta(x) = \sum_{(x)} x_{(1)} \otimes x_{(2)}$.

Case 1: *q* is not a root of unity

One important type of representation is a weight representation, and the corresponding module is called a weight module. A weight module is a module with a basis of weight vectors. A weight vector is a nonzero vector *v* such that $k_\lambda.v = d_\lambda v$ for all λ, where d_λ are complex numbers for all weights λ such that

- $d_0 = 1$,
- $d_\lambda d_\mu = d_{\lambda+\mu}$, for all weights λ and μ.

A weight module is called integrable if the actions of e_i and f_i are locally nilpotent (*i.e.* for any vector *v* in the module, there exists a positive integer *k*, possibly dependent on *v*, such that $e_i^k.v = f_i^k.v = 0$ for all *i*). In the case of integrable modules, the complex numbers d_λ associated with a weight vector satisfy $d_\lambda = c_\lambda q^{(\lambda,\nu)}$, where ν is an element of the weight lattice, and c_λ are complex numbers such that

- $c_0 = 1$,
- $c_\lambda c_\mu = c_{\lambda+\mu}$, for all weights λ and μ,
- $c_{2\alpha_i} = 1$ for all *i*.

Of special interest are highest weight representations, and the corresponding highest weight modules. A highest weight module is a module generated by a weight vector *v*, subject to $k_\lambda.v = d_\lambda v$ for all weights λ, and $e_i.v = 0$ for all *i*. Similarly, a quantum group can have a lowest weight representation and lowest weight module, *i.e.* a module generated by a weight vector *v*, subject to $k_\lambda.v = d_\lambda v$ for all weights λ, and $f_i.v = 0$ for all *i*.

Define a vector *v* to have weight ν if $k_\lambda.v = q^{(\lambda,\nu)}v$ for all λ in the weight lattice.

If *G* is a Kac-Moody algebra, then in any irreducible highest weight representation of $U_q(G)$, with highest weight ν, the multiplicities of the weights are equal to their multiplicities in an irreducible representation of $U(G)$ with equal highest weight. If the highest weight is dominant and integral (a weight μ is dominant and integral if μ satisfies the condition that $2(\mu,\alpha_i)/(\alpha_i,\alpha_i)$ is a non-negative integer for all *i*), then the weight spectrum of the irreducible representation is invariant under the Weyl group for *G*, and the representation is integrable.
Conversely, if a highest weight module is integrable, then its highest weight vector *v* satisfies $k_\lambda.v = c_\lambda q^{(\lambda,\nu)}v$, where c_λ are complex numbers such that

- $c_0 = 1$,
- $c_\lambda c_\mu = c_{\lambda+\mu}$, for all weights λ and μ,
- $c_{2\alpha_i} = 1$ for all *i*,

and ν is dominant and integral.

As is the case for all Hopf algebras, the tensor product of two modules is another module. For an element *x* of $U_q(G)$, and for vectors *v* and *w* in the respective modules, $x.(v \otimes w) = \Delta(x).(v \otimes w)$, so that $k_\lambda.(v \otimes w) = k_\lambda.v \otimes k_\lambda.w$, and in the case of coproduct Δ_1, $e_i.(v \otimes w) = k_i.v \otimes e_i.w + e_i.v \otimes w$ and $f_i.(v \otimes w) = v \otimes f_i.w + f_i.v \otimes k_i^{-1}.w$.
The integrable highest weight module described above is a tensor product of a one-dimensional module (on which $k_\lambda = c_\lambda$ for all λ, and $e_i = f_i = 0$ for all *i*) and a highest weight module generated by a nonzero vector v_0, subject to $k_\lambda.v_0 = q^{(\lambda,\nu)}v_0$ for all weights λ, and $e_i.v_0 = 0$ for all *i*.

In the specific case where *G* is a finite-dimensional Lie algebra (as a special case of a Kac-Moody algebra), then the irreducible representations with dominant integral highest weights are also finite-dimensional.

In the case of a tensor product of highest weight modules, its decomposition into submodules is the same as for the tensor product of the corresponding modules of the Kac-Moody algebra (the highest weights are the same, as are their multiplicities).

Quasitriangularity

Case 1: *q* is not a root of unity

Strictly, the quantum group $U_q(G)$ is not quasitriangular, but it can be thought of as being "nearly quasitriangular" in that there exists an infinite formal sum which plays the role of an *R*-matrix. This infinite formal sum is expressible in terms of generators e_i and f_i, and Cartan generators t_λ, where k_λ is formally identified with q^{t_λ}. The infinite formal sum is the product of two factors, $q^{\eta \sum_j t_{\lambda_j} \otimes t_{\mu_j}}$, and an infinite formal sum, where $\{\lambda_j\}$ is a basis for the dual space to the Cartan subalgebra, and $\{\mu_j\}$ is the dual basis, and η is a sign (+1 or -1).
The formal infinite sum which plays the part of the *R*-matrix has a well-defined action on the tensor product of two irreducible highest weight modules, and also on the tensor product if two lowest weight modules. Specifically, if *v* has weight α and *w* has weight β, then $q^{\eta \sum_j t_{\lambda_j} \otimes t_{\mu_j}}.(v \otimes w) = q^{\eta(\alpha,\beta)} v \otimes w$, and the fact that the modules are both highest weight modules or both lowest weight modules reduces the action of the other factor on $v \otimes w$ to a finite sum.
Specifically, if *V* is a highest weight module, then the formal infinite sum, *R*, has a well-defined, and invertible, action on $V \otimes V$, and this value of *R* (as an element of $\mathrm{Hom}(V) \otimes \mathrm{Hom}(V)$) satisfies the Yang-Baxter equation, and therefore allows us to determine a representation of the braid group, and to define quasi-invariants for knots, links and braids.

Quantum groups at *q* = 0

Masaki Kashiwara has researched the limiting behaviour of quantum groups as $q \to 0$.

As a consequence of the defining relations for the quantum group $U_q(G)$, $U_q(G)$ can be regarded as a Hopf algebra over $\mathbb{Q}(q)$, the field of all rational functions of an indeterminate *q* over $\mathbb{Q}$.
For simple root α_i and non-negative integer n, define $e_i^{(n)} = e_i^n/[n]_{q_i}!$ and $f_i^{(n)} = f_i^n/[n]_{q_i}!$ (specifically, $e_i^{(0)} = f_i^{(0)} = 1$). In an integrable module M, and for weight λ, a vector $u \in M_\lambda$ (*i.e.* a vector u in M with weight λ) can be uniquely decomposed into the sums

- $u = \sum_{n=0}^{\infty} f_i^{(n)} u_n = \sum_{n=0}^{\infty} e_i^{(n)} v_n,$

where $u_n \in \ker(e_i) \cap M_{\lambda + n\alpha_i}$, $v_n \in \ker(f_i) \cap M_{\lambda - n\alpha_i}$, $u_n \neq 0$ only if $n + \frac{2(\lambda, \alpha_i)}{(\alpha_i, \alpha_i)} \geq 0$, and $v_n \neq 0$ only if $n - \frac{2(\lambda, \alpha_i)}{(\alpha_i, \alpha_i)} \geq 0$. Linear mappings $\tilde{e}_i : M \to M$ and $\tilde{f}_i : M \to M$ can be defined on M_λ by

- $\tilde{e}_i u = \sum_{n=1}^{\infty} f_i^{(n-1)} u_n = \sum_{n=0}^{\infty} e_i^{(n+1)} v_n,$
- $\tilde{f}_i u = \sum_{n=0}^{\infty} f_i^{(n+1)} u_n = \sum_{n=1}^{\infty} e_i^{(n-1)}.v_n$

Let A be the integral domain of all rational functions in $\mathbb{Q}(q)$ which are regular at $q = 0$ (*i.e.* a rational function $f(q)$ is an element of A if and only if there exist polynomials $g(q)$ and $h(q)$ in the polynomial ring $\mathbb{Q}[q]$ such that $h(0) \neq 0$, and $f(q) = g(q)/h(q)$). A **crystal base** for M is an ordered pair (L, B), such that

- L is a free A -submodule of M such that $M = \mathbb{Q}(q) \otimes_A L$;
- B is a $\mathbb{Q}$ -basis of the vector space L/qL over $\mathbb{Q}$,
- $L = \oplus_\lambda L_\lambda$ and $B = \sqcup_\lambda B_\lambda$, where $L_\lambda = L \cap M_\lambda$ and $B_\lambda = B \cap (L_\lambda/qL_\lambda)$,
- $\tilde{e}_i L \subset L$ and $\tilde{f}_i L \subset L$ for all i,

- $\tilde{e}_i B \subset B \cup \{0\}$ and $\tilde{f}_i B \subset B \cup \{0\}$ for all i,
- for all $b \in B$ and $b' \in B$, and for all i, $\tilde{e}_i b = b'$ if and only if $\tilde{f}_i b' = b$.

To put this into a more informal setting, the actions of $e_i f_i$ and $f_i e_i$ are generally singular at $q = 0$ on an integrable module M. The linear mappings $\tilde{e}_i$ and $\tilde{f}_i$ on the module are introduced so that the actions of $\tilde{e}_i \tilde{f}_i$ and $\tilde{f}_i \tilde{e}_i$ are regular at $q = 0$ on the module. There exists a $\mathbb{Q}(q)$-basis of weight vectors $\tilde{B}$ for M, with respect to which the actions of $\tilde{e}_i$ and $\tilde{f}_i$ are regular at $q = 0$ for all *i*. The module is then restricted to the free A-module generated by the basis, and the basis vectors, the A-submodule and the actions of $\tilde{e}_i$ and $\tilde{f}_i$ are evaluated at $q = 0$. Furthermore, the basis can be chosen such that at $q = 0$, for all i, $\tilde{e}_i$ and $\tilde{f}_i$ are represented by mutual transposes, and map basis vectors to basis vectors or 0.

A crystal base can be represented by a directed graph with labelled edges. Each vertex of the graph represents an element of the $\mathbb{Q}$-basis B of L/qL, and a directed edge, labelled by *i*, and directed from vertex v_1 to vertex v_2, represents that $b_2 = \tilde{f}_i b_1$ (and, equivalently, that $b_1 = \tilde{e}_i b_2$), where b_1 is the basis element represented by v_1, and b_2 is the basis element represented by v_2. The graph completely determines the actions of $\tilde{e}_i$ and $\tilde{f}_i$ at $q = 0$. If an integrable module has a crystal base, then the module is irreducible if and only if the graph representing the crystal base is connected (a graph is called "connected" if the set of vertices cannot be partitioned into the union of nontrivial disjoint subsets V_1 and V_2 such that there are no edges joining any vertex in V_1 to any vertex in V_2).

For any integrable module with a crystal base, the weight spectrum for the crystal base is the same as the weight spectrum for the module, and therefore the weight spectrum for the crystal base is the same as the weight spectrum for the corresponding module of the appropriate Kac-Moody algebra. The multiplicities of the weights in the crystal base are also the same as their multiplicities in the corresponding module of the appropriate Kac-Moody algebra.

It is a theorem of Kashiwara that every integrable highest weight module has a crystal base. Similarly, every integrable lowest weight module has a crystal base.

Tensor products of crystal bases

Let M be an integrable module with crystal base (L, B) and M' be an integrable module with crystal base (L', B'). For crystal bases, the coproduct Δ, given by $\Delta(k_\lambda) = k_\lambda \otimes k_\lambda,\ \Delta(e_i) = e_i \otimes k_i^{-1} + 1 \otimes e_i,\ \Delta(f_i) = f_i \otimes 1 + k_i \otimes f_i$, is adopted. The integrable module $M \otimes_{\mathbb{Q}(q)} M'$ has crystal base $(L \otimes_A L', B \otimes B')$, where $B \otimes B' = \{b \otimes_{\mathbb{Q}} b' : b \in B,\ b' \in B'\}$. For a basis vector $b \in B$, define $\epsilon_i(b) = \max\{n \geq 0 : \tilde{e}_i^n b \neq 0\}$ and $\phi_i(b) = \max\{n \geq 0 : \tilde{f}_i^n b \neq 0\}$. The actions of $\tilde{e}_i$ and $\tilde{f}_i$ on $b \otimes b'$

- $\tilde{e}_i(b \otimes b') = \begin{cases} \tilde{e}_i b \otimes b', & \text{if } \phi_i(b) \geq \epsilon_i(b'), \\ b \otimes \tilde{e}_i b', & \text{if } \phi_i(b) < \epsilon_i(b'), \end{cases}$
- $\tilde{f}_i(b \otimes b') = \begin{cases} \tilde{f}_i b \otimes b', & \text{if } \phi_i(b) > \epsilon_i(b'), \\ b \otimes \tilde{f}_i b', & \text{if } \phi_i(b) \leq \epsilon_i(b'). \end{cases}$

The decomposition of the product two integrable highest weight modules into irreducible submodules is determined by the decomposition of the graph of the crystal base into its connected components (*i.e.* the highest weights of the submodules are determined, and the multiplicity of each highest weight is determined).

Compact matrix quantum groups

See also compact quantum group.

S.L. Woronowicz introduced compact matrix quantum groups. Compact matrix quantum groups are abstract structures on which the "continuous functions" on the structure are given by elements of a C*-algebra. The geometry of a compact matrix quantum group is a special case of a noncommutative geometry.

The continuous complex-valued functions on a compact Hausdorff topological space form a commutative C*-algebra. By the Gelfand theorem, a commutative C*-algebra is isomorphic to the C*-algebra of continuous complex-valued functions on a compact Hausdorff topological space, and the topological space is uniquely determined by the C*-algebra up to homeomorphism.

For a compact topological group, *G*, there exists a C*-algebra homomorphism $\Delta : C(G) \to C(G) \otimes C(G)$ (where $C(G) \otimes C(G)$ is the C*-algebra tensor product - the completion of the algebraic tensor product of $C(G)$ and $C(G)$), such that $\Delta(f)(x,y) = f(xy)$ for all $f \in C(G)$, and for all $x, y \in G$ (where $(f \otimes g)(x,y) = f(x)g(y)$ for all $f, g \in C(G)$ and all $x, y \in G$). There also exists a linear multiplicative mapping $\kappa : C(G) \to C(G)$, such that $\kappa(f)(x) = f(x^{-1})$ for all $f \in C(G)$ and all $x \in G$. Strictly, this does not make $C(G)$ a Hopf algebra, unless *G* is finite. On the other hand, a finite-dimensional representation of *G* can be used to generate a *-subalgebra of $C(G)$ which is also a Hopf *-algebra. Specifically, if $g \mapsto (u_{ij}(g))_{i,j}$ is an n -dimensional representation of G , then $u_{ij} \in C(G)$ for all i, j , and $\Delta(u_{ij}) = \sum_k u_{ik} \otimes u_{kj}$ for all i, j . It follows that the *-algebra generated by u_{ij} for all i, j and $\kappa(u_{ij})$ for all i, j is a Hopf *-algebra: the counit is determined by $\epsilon(u_{ij}) = \delta_{ij}$ for all i, j (where δ_{ij} is the Kronecker delta), the antipode is κ , and the unit is given by $1 = \sum_k u_{1k}\kappa(u_{k1}) = \sum_k \kappa(u_{1k})u_{k1}$.

As a generalization, a compact matrix quantum group is defined as a pair (C, u), where C is a C*-algebra and $u = (u_{ij})_{i,j=1,\dots,n}$ is a matrix with entries in C such that

- The *-subalgebra, C_0, of C , which is generated by the matrix elements of u , is dense in C ;
- There exists a C*-algebra homomorphism $\Delta : C \to C \otimes C$ (where $C \otimes C$ is the C*-algebra tensor product - the completion of the algebraic tensor product of C and C) such that $\Delta(u_{ij}) = \sum_k u_{ik} \otimes u_{kj}$ for all i, j (Δ is called the comultiplication);
- There exists a linear antimultiplicative map $\kappa : C_0 \to C_0$ (the coinverse) such that $\kappa(\kappa(v*)*) = v$ for all $v \in C_0$ and $\sum_k \kappa(u_{ik})u_{kj} = \sum_k u_{ik}\kappa(u_{kj}) = \delta_{ij}I$, where I is the identity element of C . Since κ is antimultiplicative, then $\kappa(vw) = \kappa(w)\kappa(v)$ for all $v, w \in C_0$.

As a consequence of continuity, the comultiplication on C is coassociative.

In general, C is not a bialgebra, and C_0 is a Hopf *-algebra.

Informally, C can be regarded as the *-algebra of continuous complex-valued functions over the compact matrix quantum group, and u can be regarded as a finite-dimensional representation of the compact matrix quantum group.

A representation of the compact matrix quantum group is given by a corepresentation of the Hopf *-algebra (a corepresentation of a counital coassociative coalgebra A is a square matrix $v = (v_{ij})_{i,j=1,\dots,n}$ with entries in A (so $v \in M_n(A)$) such that $\Delta(v_{ij}) = \sum_{k=1}^{n} v_{ik} \otimes v_{kj}$ for all i, j and $\epsilon(v_{ij}) = \delta_{ij}$ for all i, j). Furthermore, a representation *v*, is called unitary if the matrix for *v* is unitary (or equivalently, if $\kappa(v_{ij}) = v^*_{ji}$ for all *i*, *j*).

An example of a compact matrix quantum group is $SU_\mu(2)$, where the parameter μ is a positive real number. So $SU_\mu(2) = (C(SU_\mu(2), u)$, where $C(SU_\mu(2))$ is the C*-algebra generated by α and γ,subject to

$$\gamma\gamma^* = \gamma^*\gamma,\ \alpha\gamma = \mu\gamma\alpha,\ \alpha\gamma^* = \mu\gamma^*\alpha,\ \alpha\alpha^* + \mu\gamma^*\gamma = \alpha^*\alpha + \mu^{-1}\gamma^*\gamma = I,$$

and $u = \begin{pmatrix} \alpha & \gamma \\ -\gamma^* & \alpha^* \end{pmatrix}$, so that the comultiplication is determined by $\Delta(\alpha) = \alpha \otimes \alpha - \gamma \otimes \gamma^*$, $\Delta(\gamma) = \alpha \otimes \gamma + \gamma \otimes \alpha^*$, and the coinverse is determined by $\kappa(\alpha) = \alpha^*$, $\kappa(\gamma) = -\mu^{-1}\gamma$, $\kappa(\gamma^*) = -\mu\gamma^*$, $\kappa(\alpha^*) = \alpha$. Note that u is a representation, but not a unitary representation. u is equivalent to the unitary representation $v = \begin{pmatrix} \alpha & \sqrt{\mu}\gamma \\ -\frac{1}{\sqrt{\mu}}\gamma^* & \alpha^* \end{pmatrix}$.

Equivalently, $SU_\mu(2) = (C(SU_\mu(2)), w)$, where $C(SU_\mu(2))$is the C*-algebra generated by α and β ,subject to

$$\beta\beta^* = \beta^*\beta,\ \alpha\beta = \mu\beta\alpha,\ \alpha\beta^* = \mu\beta^*\alpha,\ \alpha\alpha^* + \mu^2\beta^*\beta = \alpha^*\alpha + \beta^*\beta = I,$$

and $w = \begin{pmatrix} \alpha & \mu\beta \\ -\beta^* & \alpha^* \end{pmatrix}$, so that the comultiplication is determined by $\Delta(\alpha) = \alpha \otimes \alpha - \mu\beta \otimes \beta^*$, $\Delta(\beta) = \alpha \otimes \beta + \beta \otimes \alpha^*$, and the coinverse is determined by $\kappa(\alpha) = \alpha^*$, $\kappa(\beta) = -\mu^{-1}\beta$, $\kappa(\beta^*) = -\mu\beta^*$, $\kappa(\alpha^*) = \alpha$. Note that w is a unitary representation. The realizations can be identified by equating $\gamma = \sqrt{\mu}\beta$.

When $\mu = 1$, then $SU_\mu(2)$is equal to the algebra $C(SU(2))$of functions on the concrete compact group $SU(2)$.

Bicrossproduct quantum groups

Whereas compact matrix pseudogroups are typically versions of Drinfeld-Jimbo quantum groups in a dual function algebra formulation, with additional structure, the bicrossproduct ones are a distinct second family of quantum groups of increasing importance as deformations of solvable rather than semisimple Lie groups. They are associated to Lie splittings of Lie algebras or local factorisations of Lie groups and can be viewed as the cross product or Mackey quantisation of one of the factors acting on the other for the algebra and a similar story for the coproduct Δ with the second factor acting back on the first. The very simplest nontrivial example corresponds to two copies of $\mathbb{R}$ locally acting on each other and results in a quantum group (given here in an algebraic form) with generators p, K, K^{-1}, say, and coproduct

$$[p, K] = hK(K-1),\ \Delta p = p \otimes K + 1 \otimes p,\ \Delta K = K \otimes K$$

where h is the deformation parameter. This quantum group was linked to a toy model of Planck scale physics implementing Born reciprocity when viewed as a deformation of the Heisenberg algebra of quantum mechanics. Also, starting with any compact real form of a semisimple Lie algebra g its complexification as a real Lie algebra of twice the dimension splits into g and a certain solvable Lie algebra (the Iwasawa decomposition), and this provides a canonical bicrossproduct quantum group associated to g. For $su(2)$ one obtains a quantum group deformation of the Euclidean group E(3) of motions in 3 dimensions.

Notes

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Affine quantum group

In mathematics, a **quantum affine algebra** (or **affine quantum group**) is a Hopf algebra that is a *q*-deformation of the universal enveloping algebra of an affine Lie algebra. They were introduced independently by Drinfeld (1985) and Jimbo (1985) as a special case of their general construction of a quantum group from a Cartan matrix. One of their principal applications has been to the theory of solvable lattice models in quantum statistical mechanics, where the Yang-Baxter equation occurs with a spectral parameter. Combinatorial aspects of the representation theory of quantum affine algebras can be described simply using crystal bases, which correspond to the degenerate case when the deformation parameter *q* vanishes and the Hamiltonian of the associated lattice model can be explicitly diagonalized.

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Group representation

In the mathematical field of representation theory, **group representations** describe abstract groups in terms of linear transformations of vector spaces; in particular, they can be used to represent group elements as matrices so that the group operation can be represented by matrix multiplication. Representations of groups are important because they allow many group-theoretic problems to be reduced to problems in linear algebra, which is well-understood. They are also important in physics because, for example, they describe how the symmetry group of a physical system affects the solutions of equations describing that system.

The term *representation of a group* is also used in a more general sense to mean any "description" of a group as a group of transformations of some mathematical object. More formally, a "representation" means a homomorphism from the group to the automorphism group of an object. If the object is a vector space we have a *linear representation*. Some people use *realization* for the general notion and reserve the term *representation* for the special case of linear representations. The bulk of this article describes linear representation theory; see the last section for generalizations.

Branches of group representation theory

The representation theory of groups divides into subtheories depending on the kind of group being represented. The various theories are quite different in detail, though some basic definitions and concepts are similar. The most important divisions are:

- *Finite groups* — Group representations are a very important tool in the study of finite groups. They also arise in the applications of finite group theory to crystallography and to geometry. If the field of scalars of the vector space has characteristic *p*, and if *p* divides the order of the group, then this is called *modular representation theory*; this special case has very different properties. See Representation theory of finite groups.
- *Compact groups or locally compact groups* — Many of the results of finite group representation theory are proved by averaging over the group. These proofs can be carried over to infinite groups by replacement of the average with an integral, provided that an acceptable notion of integral can be defined. This can be done for locally compact groups, using Haar measure. The resulting theory is a central part of harmonic analysis. The Pontryagin duality describes the theory for commutative groups, as a generalised Fourier transform. See also: Peter-Weyl theorem.
- *Lie groups* — Many important Lie groups are compact, so the results of compact representation theory apply to them. Other techniques specific to Lie groups are used as well. Most of the groups important in physics and chemistry are Lie groups, and their representation theory is crucial to the application of group theory in those fields. See Representations of Lie groups and Representations of Lie algebras.
- *Linear algebraic groups* (or more generally *affine group schemes*) — These are the analogues of Lie groups, but over more general fields than just **R** or **C**. Although linear algebraic groups have a classification that is very similar to that of Lie groups, and give rise to the same families of Lie algebras, their representations are rather different (and much less well understood). The analytic techniques used for studying Lie groups must be replaced by techniques from algebraic geometry, where the relatively weak Zariski topology causes many technical complications.
- *Non-compact topological groups* — The class of non-compact groups is too broad to construct any general representation theory, but specific special cases have been studied, sometimes using ad hoc techniques. The *semisimple Lie groups* have a deep theory, building on the compact case. The complementary *solvable* Lie groups cannot in the same way be classified. The general theory for Lie groups deals with semidirect products of the two types, by means of general results called *Mackey theory*, which is a generalization of Wigner's classification methods.

Representation theory also depends heavily on the type of vector space on which the group acts. One distinguishes between finite-dimensional representations and infinite-dimensional ones. In the infinite-dimensional case, additional structures are important (e.g. whether or not the space is a Hilbert space, Banach space, etc.).

One must also consider the type of field over which the vector space is defined. The most important case is the field of complex numbers. The other important cases are the field of real numbers, finite fields, and fields of p-adic numbers. In general, algebraically closed fields are easier to handle than non-algebraically closed ones. The characteristic of the field is also significant; many theorems for finite groups depend on the characteristic of the field not dividing the order of the group.

Definitions

A **representation** of a group *G* on a vector space *V* over a field *K* is a group homomorphism from *G* to *GL(V)*, the general linear group on *V*. That is, a representation is a map

$$\rho\colon G \to GL(V)$$

such that

$$\rho(g_1 g_2) = \rho(g_1)\rho(g_2), \qquad \text{for all } g_1, g_2 \in G.$$

Here *V* is called the **representation space** and the dimension of *V* is called the **dimension** of the representation. It is common practice to refer to *V* itself as the representation when the homomorphism is clear from the context.

In the case where *V* is of finite dimension *n* it is common to choose a basis for *V* and identify GL(*V*) with GL (*n*, *K*) the group of *n*-by-*n* invertible matrices on the field *K*.

If *G* is a topological group and *V* is a topological vector space, a **continuous representation** of *G* on *V* is a representation ρ such that the application $\Phi : G \times V \to V$ defined by $\Phi(g, v) = \rho(g).v$ is continuous.

The **kernel** of a representation ρ of a group *G* is defined as the normal subgroup of *G* whose image under ρ is the identity transformation:

$$\ker \rho = \{g \in G \mid \rho(g) = id\}.$$

A faithful representation is one in which the homomorphism *G* → GL(*V*) is injective; in other words, one whose kernel is the trivial subgroup {*e*} consisting of just the group's identity element.

Given two *K* vector spaces *V* and *W*, two representations

$$\rho_1\colon G \to GL(V)$$

and

$$\rho_2\colon G \to GL(W)$$

are said to be **equivalent** or **isomorphic** if there exists a vector space isomorphism

$$\alpha\colon V \to W$$

so that for all *g* in *G*

$$\alpha \circ \rho_1(g) \circ \alpha^{-1} = \rho_2(g).$$

Examples

Consider the complex number $u = e^{2\pi i / 3}$ which has the property $u^3 = 1$. The cyclic group $C_3 = \{1, u, u^2\}$ has a representation ρ on $\mathbf{C}^2$ given by:

$$\rho(1) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad \rho(u) = \begin{bmatrix} 1 & 0 \\ 0 & u \end{bmatrix} \qquad \rho(u^2) = \begin{bmatrix} 1 & 0 \\ 0 & u^2 \end{bmatrix}.$$

This representation is faithful because ρ is a one-to-one map.

An isomorphic representation for C_3 is

$$\rho(1) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad \rho(u) = \begin{bmatrix} u & 0 \\ 0 & 1 \end{bmatrix} \qquad \rho(u^2) = \begin{bmatrix} u^2 & 0 \\ 0 & 1 \end{bmatrix}.$$

The group C_3 may also be faithfully represented on $\mathbf{R}^2$ by

$$\rho(1) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad \rho(u) = \begin{bmatrix} a & -b \\ b & a \end{bmatrix} \qquad \rho(u^2) = \begin{bmatrix} a & b \\ -b & a \end{bmatrix}$$

where $a = \Re(u) = -1/2$ and $b = \Im(u) = \sqrt{3}/2$.

Reducibility

A subspace W of V that is invariant under the group action is called a *subrepresentation*. If V has exactly two subrepresentations, namely the zero-dimensional subspace and V itself, then the representation is said to be *irreducible*; if it has a proper subrepresentation of nonzero dimension, the representation is said to be *reducible*. The representation of dimension zero is considered to be neither reducible nor irreducible, just like the number 1 is considered to be neither composite nor prime.

Under the assumption that the characteristic of the field K does not divide the size of the group, representations of finite groups can be decomposed into a direct sum of irreducible subrepresentations (see Maschke's theorem). This holds in particular for any representation of a finite group over the complex numbers, since the characteristic of the complex numbers is zero, which never divides the size of a group.

In the example above, the first two representations given are both decomposable into two 1-dimensional subrepresentations (given by span{(1,0)} and span{(0,1)}), while the third representation is irreducible.

Generalizations

Set-theoretical representations

A *set-theoretic representation* (also known as a group action or *permutation representation*) of a group G on a set X is given by a function ρ from G to X^X, the set of functions from X to X, such that for all g_1, g_2 in G and all x in X:

$$\rho(1)[x] = x$$
$$\rho(g_1 g_2)[x] = \rho(g_1)[\rho(g_2)[x]].$$

This condition and the axioms for a group imply that ρ(g) is a bijection (or permutation) for all g in G. Thus we may equivalently define a permutation representation to be a group homomorphism from G to the symmetric group S_X of X.

For more information on this topic see the article on group action.

Representations in other categories

Every group G can be viewed as a category with a single object; morphisms in this category are just the elements of G. Given an arbitrary category C, a *representation* of G in C is a functor from G to C. Such a functor selects an object X in C and a group homomorphism from G to Aut(X), the automorphism group of X.

In the case where C is $\mathbf{Vect}_K$, the category of vector spaces over a field K, this definition is equivalent to a linear representation. Likewise, a set-theoretic representation is just a representation of G in the category of sets.

When C is **Ab**, the category of abelian groups, the objects obtained are called G-modules.

For another example consider the category of topological spaces, **Top**. Representations in **Top** are homomorphisms from G to the homeomorphism group of a topological space X.

Two types of representations closely related to linear representations are:

- projective representations: in the category of projective spaces. These can be described as "linear representations up to scalar transformations".
- affine representations: in the category of affine spaces. For example, the Euclidean group acts affinely upon Euclidean space.

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Unitary representation

In mathematics, a **unitary representation** of a group G is a linear representation π of G on a complex Hilbert space V such that $\pi(g)$ is a unitary operator for every $g \in G$. The general theory is well-developed in case G is a locally compact (Hausdorff) topological group and the representations are strongly continuous.

The theory has been widely applied in quantum mechanics since the 1920s, particularly influenced by Hermann Weyl's 1928 book *Gruppentheorie und Quantenmechanik*. One of the pioneers in constructing a general theory of unitary representations, for any group G rather than just for particular groups useful in applications, was George Mackey.

Context in harmonic analysis

The theory of unitary representations of groups is closely connected with harmonic analysis. In the case of an abelian group G, a fairly complete picture of the representation theory of G is given by Pontryagin duality. In general, the unitary equivalence classes of irreducible unitary representations of G make up its **unitary dual**. This set can be identified with the spectrum of the C*-algebra associated to G by the group C*-algebra construction. This is a topological space.

The general form of the Plancherel theorem tries to describe the regular representation of G on $L^2(G)$ by means of a measure on the unitary dual. For G abelian this is given by the Pontryagin duality theory. For G compact, this is done by the Peter-Weyl theorem; in that case the unitary dual is a discrete space, and the measure attaches an atom to each point of mass equal to its degree.

Formal definitions

Let G be a topological group. A **strongly continuous unitary representation** of G on a Hilbert space H is a group homomorphism from G into the unitary group of H,

$$\pi : G \to \mathrm{U}(H)$$

such that $g \to \pi(g)\,\xi$ is a norm continuous function for every $\xi \in H$.

Note that if G is a Lie group, the Hilbert space also admits underlying smooth and analytic structures. A vector ξ in H is said to be **smooth** or **analytic** if the map $g \to \pi(g)\,\xi$ is smooth or analytic (in the norm or weak topologies on H).[1] Smooth vectors are dense in H by a classical argument of Lars Gårding, since convolution by smooth functions of compact support yields smooth vectors. Analytic vectors are dense by a classical argument of Edward Nelson, amplified by Roe Goodman, since vectors in the image of a heat operator e^{-tD}, corresponding to an elliptic differential operator D in the universal enveloping algebra of G, are analytic. Not only do smooth or analytic vectors form dense subspaces; they also form common cores for the unbounded skew-adjoint operators corresponding to the elements of the Lie algebra, in the sense of spectral theory.[2]

Complete reducibility

A unitary representation is completely reducible, in the sense that for any closed invariant subspace, the orthogonal complement is again a closed invariant subspace. This is at the level of an observation, but is a fundamental property. For example, it implies that finite dimensional unitary representations are always a direct sum of irreducible representations, in the algebraic sense.

Since unitary representations are much easier to handle than the general case, it is natural to consider **unitarizable representations**, those that become unitary on the introduction of a suitable complex Hilbert space structure. This works very well for finite groups, and more generally for compact groups, by an averaging argument applied to an arbitrary hermitian structure. For example, a natural proof of Maschke's theorem is by this route.

Unitarizability and the unitary dual question

In general, for non-compact groups, it is a more serious question which representations are unitarizable. One of the important unsolved problems in mathematics is the description of the **unitary dual**, the effective classification of irreducible unitary representations of all real reductive Lie groups. All irreducible unitary representations are admissible (or rather their Harish-Chandra modules are), and the admissible representations are given by the Langlands classification, and it is easy to tell which of them have a non-trivial invariant sesquilinear form. The problem is that it is in general hard to tell when this form is positive definite. For many reductive Lie groups this has been solved; see representation theory of SL2(R) and representation theory of the Lorentz group for examples.

Notes

[1] Warner (1972)
[2] Reed and Simon (1975)

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Representation theory of the Lorentz group

The Lorentz group of theoretical physics has a variety of representations, corresponding to particles with integer and half-integer spins in quantum field theory. These representations are normally constructed out of spinors.

The group may also be represented in terms of a set of functions defined on the Riemann sphere. these are the Riemann P-functions, which are expressible as hypergeometric functions. An important special case is the subgroup SO(3), where these reduce to the spherical harmonics, and find practical application in the theory of atomic spectra.

The Lorentz group has no unitary representation of finite dimension, except for the trivial representation (where every group element is represented by 1).

Finding representations

According to general representation theory of Lie groups, one first looks for the representations of the complexification of the Lie algebra of the Lorentz group. A convenient basis for the Lie algebra of the Lorentz group is given by the three generators of rotations $J^k=\varepsilon^{ijk}L_{ij}$ and the three generators of boosts $K^i=L_{it}$ where i, j, and k run over the three spatial coordinates and ε is the Levi-Civita symbol for a three dimensional spatial slice of Minkowski space. Note that the three generators of rotations transform like components of a pseudovector **J** and the three generators of boosts transform like components of a vector **K** under the adjoint action of the spatial rotation subgroup.

This motivates the following construction: first complexify, and then change basis to the components of **A** = (**J** + i **K**)/2 and **B** = (**J** − i **K**)/2. In this basis, one checks that the components of **A** and **B** satisfy separately the commutation relations of the Lie algebra $\mathbf{sl}_2$ and moreover that they commute with each other. In other words, one has the isomorphism

$$\mathfrak{so}(3,1)\otimes\mathbb{C}\cong\mathfrak{sl}_2(\mathbb{C})\oplus\mathfrak{sl}_2(\mathbb{C}).$$

The utility of this isomorphism comes from the fact that $\mathbf{sl}_2$ is the complexification of the rotation algebra, and so its irreducible representations correspond to the well-known representations of the spatial rotation group; for each j in ½**Z**, one has the (2j+1)-dimensional spin-j representation spanned by the spherical harmonics with j as the highest weight. Thus the finite dimensional irreducible representations of the Lorentz group are simply given by an ordered pair of half-integers (m,n) which fix representations of the subalgebra spanned by the components of **A** and **B** respectively.

Properties of the (m,n) irrep

Since the angular momentum operator is given by **J** = **A** + **B**, the highest weight of the rotation subrepresentation will be m+n. So for example, the (1/2,1/2) representation has spin 1. The (m,n) representation is (2m+1)(2n+1)-dimensional.

Common representations

- (0,0) the Lorentz scalar representation. This representation is carried by relativistic scalar field theories.
- (1/2,0) is the left-handed Weyl spinor and (0,1/2) is the right-handed Weyl spinor representation.
- (1/2,0) ⊕ (0,1/2) is the bispinor representation (see also Dirac spinor).
- (1/2,1/2) is the four-vector representation. The electromagnetic vector potential lives in this rep. It is a 1-form field.
- (1,0) is the self-dual 2-form field representation and (0,1) is the anti-self-dual 2-form field representation.
- (1,0) ⊕ (0,1) is the representation of a parity invariant 2-form field. The electromagnetic field tensor transforms under this representation.

- (1,1/2) ⊕ (1/2,1) is the Rarita-Schwinger field representation.
- (1,1) is the spin-2 representation of the traceless metric tensor.

Full Lorentz group

The (*m*,*n*) representation is irreducible under the restricted Lorentz group (the identity component of the Lorentz group). When one considers the full Lorentz group, which is generated by the restricted Lorentz group along with time and parity reversal, not only is this not an irreducible representation, it is not a representation at all, unless *m*=*n*. The reason is that this representation is formed in terms of the sum of a vector and a pseudovector, and a parity reversal changes the sign of one, but not the other. The upshot is that a vector in the (*m*,*n*) representation is carried into the (*n*,*m*) representation by a parity reversal. Thus (*m*,*n*)⊕(*n*,*m*) gives an irrep of the full Lorentz group. When constructing theories such as QED which is invariant under parity reversal, Dirac spinors may be used, while theories that do not, such as the electroweak force, must be formulated in terms of Weyl spinors.

Infinite dimensional unitary representations

History

The Lorentz group SO(3,1) and its double cover SL(2,**C**) also have infinite dimensional unitary representations, first studied independently by Bargmann (1947), Gelfand & Naimark (1947) and Harish-Chandra (1947) (at the instigation of Paul Dirac). The Plancherel formula for these groups was first obtained by Gelfand and Naimark through involved calculations. The treatment was subsequently considerably simplified by Harish-Chandra (1951) and Gelfand & Graev (1953), based on an analogue for SL(2,**C**) of the integration formula of Hermann Weyl for compact Lie groups. Elementary accounts of this approach can be found in Rühl (1970) and Knapp (2001).

The theory of spherical functions for the Lorentz group, required for harmonic analysis on the 3-dimensional hyperboloid in Minkowski space, or equivalently 3-dimensional hyperbolic space, is considerably easier than the general theory. It only involves representations from the spherical principal series and can be treated directly, because in radial coordinates the Laplacian on the hyperboloid is equivalent to the Laplacian on **R**. This theory is discussed in Takahashi (1963), Helgason (1968), Helgason (2000) and the posthumous text of Jorgenson & Lang (2008).

Principal series

The **principal series**, or **unitary principal series**, are the unitary representations induced from the one-dimensional representations of the lower triangular subgroup *B* of *G* = SL(2,**C**). Since the one-dimensional representations of *B* correspond to the representations of the diagonal matrices, with non-zero complex entries *z* and z^{-1}, and thus have the form

$$\chi_{\nu,k}(re^{i\theta}) = r^{i\nu}e^{ik\theta},$$

for *k* an integer and ν real. The representations are irreducible; the only repetitions occur when *k* is replaced by −*k*. By definition the representations are realised on L^2 sections of line bundles on *G* / *B* = S^2, the Riemann sphere. When *k* = 0, these representations constitute the so-called **spherical principal series**.

The restriction of a principal series to the maximal compact subgroup *K* = SU(2) of *G* can also be realised as an induced representation of *K* using the identification *G* / *B* = *K* / *T*, where *T* = *B* ∩ *K* is the maximal torus in *K* consisting of diagonal matrices with |*z*|=1. It is the representation induced from the 1—dimensional representation z^k *T*, and is independent of ν. By Frobenius reciprocity, on *K* they decompose as a direct sum of the irreducible representations of *K* with dimensions |*k*| + 2*m* +1 with *m* a non-negative integer.

Using the identification between the Riemann sphere minus a point and **C**, the principal series can be defined directly on L^2(**C**) by the formula

$$\pi_{\nu,k}\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} f(z) = |cz+d|^{-2-i\nu}\left(\frac{cz+d}{|cz+d|}\right)^{-k} f\left(\frac{az+b}{cz+d}\right).$$

Irreducibility can be checked in a variety of ways:

- The representation is already irreducible on *B*. This can be seen directly, but is also a special case of general results on ireducibility of induced representations due to François Bruhat and George Mackey, relying on the Bruhat decomposition $G = B \cup B\, s\, B$ where *s* is the Weyl group element $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$.[1]
- The action of the Lie algebra $\mathfrak{g}$ of *G* can be computed on the algebraic direct sum of the irreducible subspaces of *K* can be computed explicitly and the it can be verified directly that the lowest dimensional subspace generates this direct sum as a $\mathfrak{g}$-module.[2] [3]

Complementary series

The for 0 < t < 2, the **complementary series** is defined on L^2 functions *f* on **C** for the inner product

$$(f,g) = \int\int \frac{f(z)\overline{g(w)}\,dz\,dw}{|z-w|^{2-t}}.$$

with the action given by

$$\pi_t\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} f(z) = |cz+d|^{-2-t} f\left(\frac{az+b}{cz+d}\right).$$

The complementary series are irreducible and inequivalent. As a representation of *K*, each is isomorphic to the Hilbert space direct sum of all the odd dimensional irreducible representations of *K* = SU(2). Irreducibility can be proved by analysing the action of $\mathfrak{g}$ on the algebraic sum of these subspaces[2] [3] or directly without using the Lie algebra.[4] [5]

Plancherel theorem

The only irreducible unitary representations of SL(2,**C**) are the principal series, the complementary series and the trivial representation. Since −*I* acts $(-1)^k$ on the principal series and trivially on the remainder, these will give all the irreducible unitary representations of the Lorentz group, provided *k* is taken to be even.

To decompose the left regular representation of *G* on $L^2(G)$, only the principal series are required. This immediately yields the decomposition on the subrepresentations $L^2(G/\pm I)$, the left regular representation of the Lorentz group, and $L^2(G / K)$, the regular representation on 3-dimensional hyperbolic space. (The former only involves principal series representations with *k* even and the latter only those with *k* = 0.)

The left and right regular representation λ and ρ are defined on $L^2(G)$ by

$$\lambda(g)f(x) = f(g^{-1}x),\ \ \rho(g)f(x) = f(xg).$$

Now if *f* is an element of $C_c(G)$, the operator π\nu,k (f) defined by

$$\pi_{\nu,k}(f) = \int_G f(g)\pi(g)\,dg$$

is Hilbert-Schmidt. We define a Hilbert space *H* by

$$H = \bigoplus_{k\geq 0} HS(L^2(C)) \otimes L^2(R, c_k(\nu^2+k^2)^{1/2}d\nu),$$

where

$$c_0 = 1/4\pi^{3/2},\ \ c_k = 1/(2\pi)^{3/2}\ \ (k \neq 0)$$

and $HS(L^2(\mathbf{C}))$ denotes the Hilbert space of Hilbert-Schmidt operators on $L^2(\mathbf{C})$.[6] Then the map *U* defined on $C_c(G)$ by

$$U(f)(\nu,k) = \pi_{\nu,k}(f)$$

extends to a unitary of $L^2(G)$ onto H.

The map U satisfies

$$U(\lambda(x)\rho(y)f)(\nu,k) = \pi_{\nu,k}(x)^{-1}\pi_{\nu,k}(f)\pi_{\nu,k}(y).$$

If f_1, f_2 are in $C_c(G)$ then

$$(f_1,f_2) = \sum_{k\geq 0} c_k^2 \int_{-\infty}^{\infty} \mathrm{Tr}(\pi_{\nu,k}(f_1)\pi_{\nu,k}(f_2)^*)(\nu^2+k^2)\,d\nu.$$

Thus if $f = f_1 \star f_2^*$ denotes the convolution of f_1 and f_2^*, where $f_2^*(g) = \overline{f_2(g^{-1})}$, then

$$f(1) = \sum_{k\geq 0} c_k^2 \int_{-\infty}^{\infty} \mathrm{Tr}(\pi_{\nu,k}(f))(\nu^2+k^2)\,d\nu.$$

The last two displayed formulas are usually referred to as the **Plancherel formula** and the **Fourier inversion formula** respectively. The Plancherel formula extends to all f_i in $L_2(G)$. By a theorem of Jacques Dixmier and Paul Malliavin, every function f in $C_c^\infty(G)$ is a finite sum of convolutions of similar functions, the inversion formula holds for such f. It can be extended to much wider classes of functions satisfying mild differentiability conditions.[7]

See also

- Poincaré group
- Wigner's classification

Notes

[1] Knapp 2001, Chapter II
[2] Harish-Chandra 1947
[3] Taylor 1986
[4] Gelfand & Naimark 1947
[5] Takahashi 1963, p. 343
[6] Note that for a Hilbert space H, HS(H) may be identified canonically with the Hilbert space tensor product of H and its conjugate space.
[7] Knapp 2001

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Stone–von Neumann theorem

In mathematics and in theoretical physics, the **Stone–von Neumann theorem** is any one of a number of different formulations of the uniqueness of the canonical commutation relations between position and momentum operators. The name is for Marshall Stone and von Neumann (1931).

Trying to represent the commutation relations

In quantum mechanics, physical observables are represented mathematically by linear operators on Hilbert spaces. For a single particle moving on the real line **R**, there are two important observables: position and momentum. In the quantum-mechanical description of such a particle, the position operator *Q* and momentum operator *P* are respectively given by

$$[Q\psi](x) = x\psi(x)$$
$$[P\psi](x) = \frac{\hbar}{i}\psi'(x)$$

on the domain *V* of infinitely differentiable functions of compact support on **R**. We assume $\hbar$ is a fixed *non-zero* real number — in quantum theory $\hbar$ is (up to a factor of 2π) Planck's constant, which is not dimensionless; it takes a small numerical value in terms of units in the macroscopic world. The operators *P*, *Q* satisfy the commutation relation

$$QP - PQ = -\frac{\hbar}{i}\mathbf{1}.$$

Already in his classic volume, Hermann Weyl observed that this commutation law was impossible for linear operators *P*, *Q* acting on finite dimensional spaces (as is clear by applying the trace of a matrix), unless $\hbar$ vanishes. A little analysis shows that in fact any two self-adjoint operators satisfying the above commutation relation cannot be both bounded.

Uniqueness of representation

One would like to classify representations of the canonical commutation relation by two self-adjoint operators acting on separable Hilbert spaces, up to unitary equivalence. By Stone's theorem, there is a one-to-one correspondence between self-adjoint operators and (strongly continuous) one parameter unitary groups. Let *Q* and *P* be two self-adjoint operators satisfying the canonical commutation relation, and e^{itQ} and e^{isP} be the corresponding unitary groups given by functional calculus. A formal computation with power series shows that

$$e^{itQ}e^{isP} - e^{ist}e^{isP}e^{itQ} = 0.$$

Conversely, given two one parameter unitary groups $U(t)$ and $V(s)$ satisfying the relation

$$U(t)V(s) = e^{ist}V(s)U(t) \quad \forall s, t, \qquad (*)$$

formally differentiating at 0 shows that the two infinitesmal generators satisfy the canonical commutation relation. These formal calculations can be made rigorous. Therefore there is a one-to-one correspondence between representations of the canonical commutation relation and two one parameter unitary groups $U(t)$ and $V(s)$ satisfying (*). This formulation of the canonical commutation relations for one parameter unitary groups is called the **Weyl form** of the CCR.

The problem now thus becomes classifying two jointly irreducible one parameter unitary groups $U(t)$ and $V(s)$ satisfying the Weyl relation on separable Hilbert spaces. The answer is the content of the **Stone-von Neumann theorem**: all such pairs of one parameter unitary groups are unitarily equivalent. In other words, for any two such $U(t)$ and $V(s)$ acting jointly irreducibly on a Hilbert space H, there is a unitary operator

$$W : L^2(\mathbb{R}) \to H$$

so that

$$W^*U(t)W = e^{isQ} \quad \text{and} \quad W^*V(t)W = e^{isP},$$

where P and Q are the position and momentum operators from above.

Historically this result was significant because it was a key step in proving that Heisenberg's matrix mechanics which presents quantum mechanical observables and dynamics in terms of infinite matrices, is unitarily equivalent to Schrödinger's wave mechanical formulation (see Schrödinger picture).

Representation theory formulation

In terms of representation theory, the Stone–von Neumann theorem classifies certain unitary representations of the Heisenberg group. This is discussed in more detail in the Heisenberg group section, below.

Informally stated, with certain technical assumptions, every representation of the Heisenberg group H_{2n+1} is equivalent to the position operators and momentum operators on $\mathbf{R}^n$. Alternatively, that they are all equivalent to the Weyl algebra (or CCR algebra) on a symplectic space of dimension $2n$.

More formally, there is a unique (up to scale) non-trivial central strongly continuous unitary representation.

This was later generalized by Mackey theory – and was the motivation for the introduction of the Heisenberg group in quantum physics.

In detail:

- The continuous Heisenberg group is a central extension of the abelian Lie group $\mathbf{R}^{2n}$ by a copy of $\mathbf{R}$,
- the corresponding Heisenberg algebra is a central extension of the abelian Lie algebra $\mathbf{R}^{2n}$ (with trivial bracket) by a copy of $\mathbf{R}$,
- the discrete Heisenberg group is a central extension of the free abelian group $\mathbf{Z}^{2n}$ by a copy of $\mathbf{Z}$, and
- the discrete Heisenberg group module p is a central extension of the free abelian p-group $(\mathbf{Z}/p\mathbf{Z})^{2n}$ by a copy of $\mathbf{Z}/p\mathbf{Z}$.

These are thus all semidirect product, and hence relatively easily understood. In all cases, if one has a representation $H \to A$ where the center maps to zero, then one simply has a representation of the corresponding abelian group or algebra, which is Fourier theory.

If the center does not map to zero, one has a more interesting theory, particularly if one restricts oneself to *central* representations.

Concretely, by a central representation one means a representation such that the center of the Heisenberg group maps into the center of the algebra: for example, if one is studying matrix representations or representations by operators on a Hilbert space, then the center of the matrix algebra or the operator algebra is the scalar matrices. Thus the representation of the center of the Heisenberg group is determined by a scale value, called the **quantization** value (in physics terms, Planck's constant), and if this goes to zero, one gets a representation of the abelian group (in physics terms, this is the classical limit).

More formally, the group algebra of the Heisenberg group $K[H]$ has center $K[\mathbf{R}]$, so rather than simply thinking of the group algebra as an algebra over the field of scalars *K*, one may think of it as an algebra over the commutative algebra $K[\mathbf{R}]$. As the center of a matrix algebra or operator algebra is the scalar matrices, a $K[\mathbf{R}]$-structure on the matrix algebra is a choice of scalar matrix – a choice of scale. Given such a choice of scale, a central representation of the Heisenberg group is a map of $K[\mathbf{R}]$-algebras $K[H] \to A$, which is the formal way of saying that it sends the center to a chosen scale.

Then the Stone–von Neumann theorem is that, given a quantization value, every strongly continuous unitary representation is unitarily equivalent to the standard representation as position and momentum.

Reformulation via Fourier transform

Let G be a locally compact abelian group and $G^\wedge$ be the Pontryagin dual of G. The Fourier-Plancherel transform defined by

$$f \mapsto \hat{f}(\gamma) = \int_G \overline{\gamma(t)} f(t) d\mu(t)$$

extends to a C*-isomorphism from the group C*-algebra C*(G) of G and $C_0(G^\wedge)$, i.e. the spectrum of C*(G) is precisely $G^\wedge$. When G is the real line **R**, this is Stone's theorem characterizing one parameter unitary groups. The theorem of Stone-von Neumann can also be restated using similar language.

The group G acts on the C*-algebra $C_0(G)$ by right translation ϱ: for s in G and f in $C_0(G)$,

$$(s \cdot f)(t) = f(t + s).$$

Under the isomorphism given above, this action becomes the natural action of G on C*($G^\wedge$):

$$\widehat{(s \cdot f)}(\gamma) = \gamma(s) \hat{f}(\gamma).$$

So a covariant representation corresponding to the C*-crossed product

$$C^*(\hat{G}) \rtimes_{\hat{\rho}} G$$

is a unitary representation $U(s)$ of G and $V(\gamma)$ of $G^\wedge$ such that

$$U(s)V(\gamma)U^*(s) = \gamma(s)V(\gamma).$$

It is a general fact that covariant representations are in one-to-one correspondence with *-representation of the corresponding crossed product. On the other hand, all irreducible representations of

$$C_0(G) \rtimes_\rho G$$

are unitarily equivalent to the $K(L(^2G))$, the compact operators on $L(^2G))$. Therefore all pairs $\{U(s), V(\gamma)\}$ are unitarily equivalent. Specializing to the case where $G = \mathbf{R}$ yields the Stone-von Neumann theorem.

The Heisenberg group

The commutation relations for P, Q look very similar to the commutation relations that define the Lie algebra of general Heisenberg group H_n for n a positive integer. This is the Lie group of $(n+2) \times (n+2)$ square matrices of the form

$$\mathrm{M}(a, b, c) = \begin{bmatrix} 1 & a & c \\ 0 & 1_n & b \\ 0 & 0 & 1 \end{bmatrix}.$$

In fact, using the Heisenberg group, we can formulate a far-reaching generalization of the Stone von Neumann theorem. Note that the center of H_n consists of matrices M(0, 0, c).

Theorem. For each non-zero real number h there is an irreducible representation U_h acting on the Hilbert space $L^2(\mathbf{R}^n)$ by

$$[U_h(\mathrm{M}(a,b,c))]\psi(x) = e^{i(b\cdot x+hc)}\psi(x+ha).$$

All these representations are unitarily inequivalent and any irreducible representation which is not trivial on the center of H_n is unitarily equivalent to exactly one of these.

Note that U_h is a unitary operator because it is the composition of two operators which are easily seen to be unitary: the translation to the *left* by $h\ a$ and multiplication by a function of absolute value 1. To show U_h is multiplicative is a straightforward calculation. The hard part of the theorem is showing the uniqueness which is beyond the scope of the article. However, below we sketch a proof of the corresponding Stone–von Neumann theorem for certain finite Heisenberg groups.

In particular, irreducible representations π, π' of the Heisenberg group H_n which are non-trivial on the center of H_n are unitarily equivalent if and only if $\pi(z) = \pi'(z)$ for any z in the center of H_n.

One representation of the Heisenberg group that is important in number theory and the theory of modular forms is the theta representation, so named because the Jacobi theta function is invariant under the action of the discrete subgroup of the Heisenberg group.

Relation to the Fourier transform

For any non-zero h, the mapping

$$\alpha_h : \mathrm{M}(a,b,c) \to \mathrm{M}(-h^{-1}b, ha, c-ab)$$

is an automorphism of H_n which is the identity on the center of H_n. In particular, the representations U_h and $U_h\,\alpha$ are unitarily equivalent. This means that there is a unitary operator W on $L^2(\mathbf{R}^n)$ such that for any g in H_n,

$$WU_h(g)W^* = U_h\alpha(g).$$

Moreover, by irreducibility of the representations U_h, it follows that up to a scalar, such an operator W is unique (cf. Schur's lemma).

Theorem. The operator W is, up to a scalar multiple, the Fourier transform on $L^2(\mathbf{R}^n)$.

This means that (ignoring the factor of $(2\pi)^{n/2}$ in the definition of the Fourier transform)

$$\int_{\mathbb{R}^n} e^{-ix\cdot p}e^{i(b\cdot x+hc)}\psi(x+ha)\ dx = e^{i(ha\cdot p+h(c-b\cdot a))}\int_{\mathbb{R}^n} e^{-iy\cdot(p-b)}\psi(y)\ dy.$$

The previous theorem can actually be used to prove the unitary nature of the Fourier transform, also known as the Plancherel theorem. Moreover, note that

$$(\alpha_h)^2\mathrm{M}(a,b,c) = \mathrm{M}(-a,-b,c).$$

Theorem. The operator W_1 such that

$$W_1U_hW_1^* = U_h\alpha^2(g)$$

is the reflection operator

$$[W_1\psi](x) = \psi(-x).$$

From this fact the Fourier inversion formula easily follows.

Representations of finite Heisenberg groups

The Heisenberg group $H_n(\mathbf{K})$ is defined for any commutative ring **K**. In this section let us specialize to the field $\mathbf{K} = \mathbf{Z}/p\,\mathbf{Z}$ for p a prime. This field has the property that there is an imbedding ω of **K** as an additive group into the circle group **T**. Note that $H_n(\mathbf{K})$ is finite with cardinality $|\mathbf{K}|^{2n+1}$. For finite Heisenberg group $H_n(\mathbf{K})$ one can give a simple proof of the Stone–von Neumann theorem using simple properties of character functions of representations. These properties follow from the orthogonality relations for characters of representations of finite groups.

For any non-zero h in **K** define the representation U_h on the finite-dimensional inner product space $l^2(\mathbf{K}^n)$ by

$$[U_h \mathrm{M}(a,b,c)\psi](x) = \omega(b\cdot x + hc)\psi(x+ha).$$

Theorem. For a fixed non-zero h, the character function χ of U_h is given by:

$$\chi(\mathrm{M}(a,b,c)) = \begin{cases} |\mathbf{K}|^n\ \omega(hc) & \text{if } a=b=0 \\ 0 & \text{otherwise.} \end{cases}$$

It follows that

$$\frac{1}{|\mathrm{H}_n(\mathbf{K})|}\sum_{g\in \mathrm{H}_n(\mathbf{K})} |\chi(g)|^2 = \frac{1}{|\mathbf{K}|^{2n+1}}|\mathbf{K}|^{2n}|\mathbf{K}| = 1.$$

By the orthogonality relations for characters of representations of finite groups this fact implies the corresponding Stone–von Neumann theorem for Heisenberg groups $H_n(\mathbf{Z}/p\,\mathbf{Z})$, particularly:

- Irreducibility of U_h
- Pairwise inequivalence of all the representations U_h.

Generalizations

The Stone–von Neumann theorem admits numerous generalizations. Much of the early work of George Mackey was directed at obtaining a formulation of the theory of induced representations developed originally by Frobenius for finite groups to the context of unitary representations of locally compact topological groups.

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Peter–Weyl theorem

In mathematics, the **Peter–Weyl theorem** is a basic result in the theory of harmonic analysis, applying to topological groups that are compact, but are not necessarily abelian. It was initially proved by Hermann Weyl, with his student Fritz Peter, in the setting of a compact topological group *G* (Peter & Weyl 1927). The theorem is a collection of results generalizing the significant facts about the decomposition of the regular representation of any finite group, as discovered by F. G. Frobenius and Issai Schur.

The theorem has three parts. The first part states that the matrix coefficients of *G* are dense in the space *C*(*G*) of continuous complex-valued functions on *G*, and thus also in the space $L^2(G)$ of square-integrable functions. The second part asserts the complete reducibility of unitary representations of *G*. The third part then asserts that the regular representation of *G* on $L^2(G)$ decomposes as the direct sum of all irreducible unitary representations. Moreover, the matrix coefficients of the irreducible unitary representations form an orthonormal basis of $L^2(G)$.

Matrix coefficients

A **matrix coefficient** of the group *G* is a complex-valued function ϕ ; on *G* given as the composition

$$\varphi = L \circ \pi$$

where $\pi : G \to \mathrm{GL}(V)$ is a finite-dimensional (continuous) group representation of *G*, and *L* is a linear functional on the vector space of endomorphisms of *V* (e.g. trace), which contains GL(*V*) as an open subset. Matrix coefficients are continuous, since representations are by definition continuous, and linear functionals on finite-dimensional spaces are also continuous.

The first part of the Peter–Weyl theorem asserts (Bump 2004, §4.1; Knapp 1986, Theorem 1.12):

- The set of matrix coefficients of *G* is dense in the space of continuous complex functions C(*G*) on *G*, equipped with the uniform norm.

This first result resembles the Stone-Weierstrass theorem in that it indicates the density of a set of functions in the space of all continuous functions, subject only to an *algebraic* characterization. In fact, if *G* is a matrix group, then the result follows easily from the Stone-Weierstrass theorem (Knapp 1986, p. 17). Conversely, it is a consequence of the subsequent conclusions of the theorem that any compact Lie group is isomorphic to a matrix group (Knapp 1986, Theorem 1.15).

A corollary of this result is that the matrix coefficients of *G* are dense in $\mathrm{L}^2(G)$.

Decomposition of a unitary representation

The second part of the theorem gives the existence of a decomposition of a unitary representation of *G* into finite-dimensional representations. Now, intuitively groups were conceived as rotations on geometric objects, so it is only natural to study representations which essentially arise from continuous **actions** on Hilbert spaces. (For those who were first introduced to dual groups consisting of characters which are the continuous homomorphisms into the circle group $\{z\in\mathbb{C}:|z|=1\}$, this approach is similar except that the circle group is (ultimately) generalised to the group of unitary operators on a given Hilbert space.)

Let *G* be a topological group and *H* a complex Hilbert space.

Given a continuous action $*:G\times H\to H$, it gives rise to a map $\rho_*:G\to^H H$ defined in the obvious way: $\rho_*(v)=gv$. This map is clearly an homomorphism from *G* into *GL(H)*, the homeomorphic automorphisms of *H*. And given such a map, we can uniquely recover the action in the obvious way.

Thus we define the **representations of *G* on an Hilbert space *H*** to be those group homomorphisms, ρ, which arise from continuous actions of *G* on *H*. We say that a representation ρ is **unitary** if ρ(*g*) is a unitary operator for all $g \in G$; i.e., $\langle gv,gw\rangle=\langle v,w\rangle$ for all $v, w \in H$. (I.e. it is unitary if $\rho:G\to U(H)$. Notice how this generalises the special case of the one-dimensional Hilbert space, where $U(\mathbb{C}^1)$ is just the circle group.)

Given these definitions, we can state the Peter–Weyl theorem. The second part of this theorem asserts (Knapp 1986, Theorem 1.14):

- Let ρ be a unitary representation of a compact group *G* on a complex Hilbert space *H*. Then *H* splits into an orthogonal direct sum of irreducible finite-dimensional unitary representations of *G*.

Decomposition of square-integrable functions

To state the third and final part of the theorem, there is a natural Hilbert space over *G* consisting of square-integrable functions, $L^2(G)$; this makes sense because Haar measure exists on *G*. Calling this Hilbert space *H*, the group *G* has a unitary representation ρ on *H* by acting on the left, via

$$\rho(h)f(g) = f(h^{-1}g).$$

The final statement of the Peter–Weyl theorem (Knapp 1986, Theorem 1.14) gives an explicit orthonormal basis of $\mathrm{L}^2(G)$. Roughly it asserts that the matrix coefficients for *G*, suitably renormalized, are an orthonormal basis of $\mathrm{L}^2(G)$. In particular, $\mathrm{L}^2(G)$ decomposes into an orthogonal direct sum of all the irreducible unitary representations, in which the multiplicity of each irreducible representation is equal to its degree (that is, the dimension of the underlying space of the representation). Thus,

$$L^2(G) = \widehat{\bigoplus_{\pi\in\Sigma}} E_\pi$$

where Σ denotes the set of (isomorphism classes of) irreducible unitary representations of *G*, and the summation denotes the closure of the direct sum of the total spaces E_π of the representations π.

More precisely, suppose that a representative π is chosen for each isomorphism class of irreducible unitary representation, and denote the collection of all such π by Σ. Let $u_{ij}^{(\pi)}$ be the matrix coefficients of π in an orthonormal basis, in other words

$$u_{ij}^{(\pi)}(g) = \langle \pi(g)e_i, e_j\rangle.$$

for each $g \in G$. Finally, let $d^{(\pi)}$ be the degree of the representation π. The theorem now asserts that the set of functions

$$\left\{\sqrt{d^{(\pi)}}u_{ij}^{(\pi)} \mid \pi\in\Sigma,\ 1\le i,j\le d^{(\pi)}\right\}$$

is an orthonormal basis of $\mathrm{L}^2(G)$.

Consequences

Structure of compact topological groups

From the theorem, one can deduce a significant general structure theorem. Let *G* be a compact topological group, which we assume Hausdorff. For any finite-dimensional *G*-invariant subspace *V* in $L^2(G)$, where *G* acts on the left, we consider the image of *G* in GL(*V*). It is closed, since *G* is compact, and a subgroup of the Lie group GL(*V*). It follows by a theorem of Élie Cartan that the image of *G* is a Lie group also.

If we now take the limit (in the sense of category theory) over all such spaces *V*, we get a result about *G* - because *G* acts faithfully on $L^2(G)$. We can say that *G* is an *inverse limit of Lie groups*. It may of course not itself be a Lie group: it may for example be a profinite group.

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[2] http://www.jstor.org/stable/1970281

Quantum algebra

Quantum algebra is one of the top-level mathematics categories used by the arXiv.

Subjects include:

- Quantum groups
- Skein theories
- Operadic algebra
- Diagrammatic algebra
- Quantum field theory

External links

- Quantum algebra at arxiv.org [1]

References

[1] http://arxiv.org/list/math.QA/current

Quantum affine algebra

In mathematics, a **quantum affine algebra** (or **affine quantum group**) is a Hopf algebra that is a q-deformation of the universal enveloping algebra of an affine Lie algebra. They were introduced independently by Drinfeld (1985) and Jimbo (1985) as a special case of their general construction of a quantum group from a Cartan matrix. One of their principal applications has been to the theory of solvable lattice models in quantum statistical mechanics, where the Yang-Baxter equation occurs with a spectral parameter. Combinatorial aspects of the representation theory of quantum affine algebras can be described simply using crystal bases, which correspond to the degenerate case when the deformation parameter q vanishes and the Hamiltonian of the associated lattice model can be explicitly diagonalized.

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Clifford algebra

In mathematics, **Clifford algebras** are a type of associative algebra. They can be thought of as one of the possible generalizations of the complex numbers and quaternions. The theory of Clifford algebras is intimately connected with the theory of quadratic forms and orthogonal transformations. Clifford algebras have important applications in a variety of fields including geometry and theoretical physics. They are named after the English geometer William Kingdon Clifford.

Introduction and basic properties

Specifically, a Clifford algebra is a unital associative algebra which contains and is generated by a vector space V equipped with a quadratic form Q. The Clifford algebra $C\ell(V,Q)$ is the "freest" algebra generated by V subject to the condition[1]

$$v^2 = Q(v)1 \quad \text{for all } v \in V.$$

If the characteristic of the ground field K is not 2, then one can rewrite this fundamental identity in the form

$$uv + vu = 2\langle u, v\rangle \text{ for all } u, v \in V,$$

where $\langle u, v\rangle = \frac{1}{2}(Q(u + v) - Q(u) - Q(v))$ is the symmetric bilinear form associated to Q, via the polarization identity. The idea of being the "freest" or "most general" algebra subject to this identity can be formally expressed through the notion of a universal property, as done below.

Quadratic forms and Clifford algebras in characteristic 2 form an exceptional case. In particular, if char $K = 2$ it is not true that a quadratic form determines a symmetric bilinear form, or that every quadratic form admits an orthogonal basis. Many of the statements in this article include the condition that the characteristic is not 2, and are false if this condition is removed.

As quantization of exterior algebra

Clifford algebras are closely related to exterior algebras. In fact, if $Q = 0$ then the Clifford algebra $C\ell(V,Q)$ is just the exterior algebra $\Lambda(V)$. For nonzero Q there exists a canonical *linear* isomorphism between $\Lambda(V)$ and $C\ell(V,Q)$ whenever the ground field K does not have characteristic two. That is, they are naturally isomorphic as vector spaces, but with different multiplications (in the case of characteristic two, they are still isomorphic as vector spaces, just not naturally). Clifford multiplication is strictly richer than the exterior product since it makes use of the extra information provided by Q.

More precisely, Clifford algebras may be thought of as *quantizations* (cf. quantization (physics), Quantum group) of the exterior algebra, in the same way that the Weyl algebra is a quantization of the symmetric algebra.

Weyl algebras and Clifford algebras admit a further structure of a *-algebra, and can be unified as even and odd terms of a superalgebra, as discussed in CCR and CAR algebras.

Universal property and construction

Let V be a vector space over a field K, and let $Q : V \to K$ be a quadratic form on V. In most cases of interest the field K is either **R**, **C** or a finite field.

A Clifford algebra $C\ell(V,Q)$ is a unital associative algebra over K together with a linear map $i : V \to C\ell(V,Q)$ satisfying $i(v)^2 = Q(v)1$ for all $v \in V$, defined by the following universal property: Given any associative algebra A over K and any linear map $j : V \to A$ such that

$$j(v)^2 = Q(v)1 \text{ for all } v \in V$$

(where 1 denotes the multiplicative identity of A), there is a unique algebra homomorphism $f : C\ell(V,Q) \to A$ such that the following diagram commutes (i.e. such that $f \circ i = j$):

$$\begin{tikzcd} V \arrow[r, "i"] \arrow[dr, "j"'] & C\ell(V,Q) \arrow[d, dashed, "f"] \\ & A \end{tikzcd}$$

Working with a symmetric bilinear form $<\cdot,\cdot>$ instead of Q (in characteristic not 2), the requirement on j is

$j(v)j(w) + j(w)j(v) = 2<v, w>$ for all $v, w \in V$.

A Clifford algebra as described above always exists and can be constructed as follows: start with the most general algebra that contains V, namely the tensor algebra $T(V)$, and then enforce the fundamental identity by taking a suitable quotient. In our case we want to take the two-sided ideal I_Q in $T(V)$ generated by all elements of the form

$v \otimes v - Q(v)1$ for all $v \in V$

and define $C\ell(V,Q)$ as the quotient

$C\ell(V,Q) = \mathrm{T}(V)/I_Q.$

It is then straightforward to show that $C\ell(V,Q)$ contains V and satisfies the above universal property, so that $C\ell$ is unique up to a unique isomorphism; thus one speaks of "the" Clifford algebra $C\ell(V, Q)$. It also follows from this construction that i is injective. One usually drops the i and considers V as a linear subspace of $C\ell(V,Q)$.

The universal characterization of the Clifford algebra shows that the construction of $C\ell(V,Q)$ is *functorial* in nature. Namely, $C\ell$ can be considered as a functor from the category of vector spaces with quadratic forms (whose morphisms are linear maps preserving the quadratic form) to the category of associative algebras. The universal property guarantees that linear maps between vector spaces (preserving the quadratic form) extend uniquely to algebra homomorphisms between the associated Clifford algebras.

Basis and dimension

If the dimension of V is n and $\{e_1,\ldots,e_n\}$ is a basis of V, then the set

$$\{e_{i_1}e_{i_2}\cdots e_{i_k} \mid 1 \le i_1 < i_2 < \cdots < i_k \le n \text{ and } 0 \le k \le n\}$$

is a basis for $C\ell(V,Q)$. The empty product ($k = 0$) is defined as the multiplicative identity element. For each value of k there are n choose k basis elements, so the total dimension of the Clifford algebra is

$$\dim C\ell(V,Q) = \sum_{k=0}^{n} \binom{n}{k} = 2^n.$$

Since V comes equipped with a quadratic form, there is a set of privileged bases for V: the orthogonal ones. An orthogonal basis is one such that

$$\langle e_i, e_j \rangle = 0 \qquad i \neq j.$$

where $<\cdot,\cdot>$ is the symmetric bilinear form associated to Q. The fundamental Clifford identity implies that for an orthogonal basis

$$e_i e_j = -e_j e_i \qquad i \neq j.$$

This makes manipulation of orthogonal basis vectors quite simple. Given a product $e_{i_1}e_{i_2}\cdots e_{i_k}$ of *distinct* orthogonal basis vectors, one can put them into standard order by including an overall sign corresponding to the number of flips needed to correctly order them (i.e. the signature of the ordering permutation).

If the characteristic is not 2 then an orthogonal basis for V exists, and one can easily extend the quadratic form on V to a quadratic form on all of $C\ell(V,Q)$ by requiring that distinct elements $e_{i_1}e_{i_2}\cdots e_{i_k}$ are orthogonal to one another whenever the $\{e_i\}$'s are orthogonal. Additionally, one sets

$$Q(e_{i_1}e_{i_2}\cdots e_{i_k}) = Q(e_{i_1})Q(e_{i_2})\cdots Q(e_{i_k}).$$

The quadratic form on a scalar is just $Q(\lambda) = \lambda^2$. Thus, orthogonal bases for V extend to orthogonal bases for $C\ell(V,Q)$. The quadratic form defined in this way is actually independent of the orthogonal basis chosen (a basis-independent formulation will be given later).

Examples: real and complex Clifford algebras

The most important Clifford algebras are those over real and complex vector spaces equipped with nondegenerate quadratic forms. The geometric interpretation of nondegenerate Clifford algebras is known as geometric algebra.

Every nondegenerate quadratic form on a finite-dimensional real vector space is equivalent to the standard diagonal form:

$$Q(v) = v_1^2 + \cdots + v_p^2 - v_{p+1}^2 - \cdots - v_{p+q}^2$$

where $n = p + q$ is the dimension of the vector space. The pair of integers (p, q) is called the signature of the quadratic form. The real vector space with this quadratic form is often denoted $\mathbf{R}^{p,q}$. The Clifford algebra on $\mathbf{R}^{p,q}$ is denoted $C\ell_{p,q}(\mathbf{R})$. The symbol $C\ell_n(\mathbf{R})$ means either $C\ell_{n,0}(\mathbf{R})$ or $C\ell_{0,n}(\mathbf{R})$ depending on whether the author prefers positive definite or negative definite spaces.

A standard orthonormal basis $\{e_i\}$ for $\mathbf{R}^{p,q}$ consists of $n = p + q$ mutually orthogonal vectors, p of which have norm +1 and q of which have norm −1. The algebra $C\ell_{p,q}(\mathbf{R})$ will therefore have p vectors which square to +1 and q vectors which square to −1.

Note that $C\ell_{0,0}(\mathbf{R})$ is naturally isomorphic to $\mathbf{R}$ since there are no nonzero vectors. $C\ell_{0,1}(\mathbf{R})$ is a two-dimensional algebra generated by a single vector e_1 which squares to −1, and therefore is isomorphic to $\mathbf{C}$, the field of complex numbers. The algebra $C\ell_{0,2}(\mathbf{R})$ is a four-dimensional algebra spanned by $\{1, e_1, e_2, e_1e_2\}$. The latter three elements square to −1 and all anticommute, and so the algebra is isomorphic to the quaternions $\mathbf{H}$. The next algebra in the sequence is $C\ell_{0,3}(\mathbf{R})$ is an 8-dimensional algebra isomorphic to the direct sum $\mathbf{H} \oplus \mathbf{H}$ called split-biquaternions.

One can also study Clifford algebras on complex vector spaces. Every nondegenerate quadratic form on a complex vector space is equivalent to the standard diagonal form

$$Q(z) = z_1^2 + z_2^2 + \cdots + z_n^2$$

where $n = \dim V$, so there is essentially only one Clifford algebra in each dimension. We will denote the Clifford algebra on $\mathbf{C}^n$ with the standard quadratic form by $C\ell_n(\mathbf{C})$. One can show that the algebra $C\ell_n(\mathbf{C})$ may be obtained as the complexification of the algebra $C\ell_{p,q}(\mathbf{R})$ where $n = p + q$:

$$C\ell_n(\mathbb{C}) \cong C\ell_{p,q}(\mathbb{R}) \otimes \mathbb{C} \cong C\ell(\mathbb{C}^{p+q}, Q \otimes \mathbb{C}).$$

Here Q is the real quadratic form of signature (p,q). Note that the complexification does not depend on the signature. The first few cases are not hard to compute. One finds that

$C\ell_0(\mathbf{C}) = \mathbf{C}$

$C\ell_1(\mathbf{C}) = \mathbf{C} \oplus \mathbf{C}$

$C\ell_2(\mathbf{C}) = M_2(\mathbf{C})$

where $M_2(\mathbf{C})$ denotes the algebra of 2×2 matrices over $\mathbf{C}$.

It turns out that every one of the algebras $C\ell_{p,q}(\mathbf{R})$ and $C\ell_n(\mathbf{C})$ is isomorphic to a matrix algebra over $\mathbf{R}$, $\mathbf{C}$, or $\mathbf{H}$ or to a direct sum of two such algebras. For a complete classification of these algebras see classification of Clifford algebras.

Properties

Relation to the exterior algebra

Given a vector space V one can construct the exterior algebra $\Lambda(V)$, whose definition is independent of any quadratic form on V. It turns out that if F does not have characteristic 2 then there is a natural isomorphism between $\Lambda(V)$ and $C\ell(V,Q)$ considered as vector spaces (and there exists an isomorphism in characteristic two, which may not be natural). This is an algebra isomorphism if and only if $Q = 0$. One can thus consider the Clifford algebra $C\ell(V,Q)$ as an enrichment (or more precisely, a quantization, cf. the Introduction) of the exterior algebra on V with a multiplication that depends on Q (one can still define the exterior product independent of Q).

The easiest way to establish the isomorphism is to choose an *orthogonal* basis $\{e_i\}$ for V and extend it to an orthogonal basis for $C\ell(V,Q)$ as described above. The map $C\ell(V,Q) \to \Lambda(V)$ is determined by

$$e_{i_1}e_{i_2}\cdots e_{i_k} \mapsto e_{i_1}\wedge e_{i_2}\wedge\cdots\wedge e_{i_k}.$$

Note that this only works if the basis $\{e_i\}$ is orthogonal. One can show that this map is independent of the choice of orthogonal basis and so gives a natural isomorphism.

If the characteristic of K is 0, one can also establish the isomorphism by antisymmetrizing. Define functions $f_k : V \times \ldots \times V \to C\ell(V,Q)$ by

$$f_k(v_1,\cdots,v_k) = \frac{1}{k!}\sum_{\sigma\in S_k}\operatorname{sgn}(\sigma)\,v_{\sigma(1)}\cdots v_{\sigma(k)}$$

where the sum is taken over the symmetric group on k elements. Since f_k is alternating it induces a unique linear map $\Lambda^k(V) \to C\ell(V,Q)$. The direct sum of these maps gives a linear map between $\Lambda(V)$ and $C\ell(V,Q)$. This map can be shown to be a linear isomorphism, and it is natural.

A more sophisticated way to view the relationship is to construct a filtration on $C\ell(V,Q)$. Recall that the tensor algebra $T(V)$ has a natural filtration: $F^0 \subset F^1 \subset F^2 \subset \ldots$ where F^k contains sums of tensors with rank $\leq k$. Projecting this down to the Clifford algebra gives a filtration on $C\ell(V,Q)$. The associated graded algebra

$$Gr_F C\ell(V,Q) = \bigoplus_k F^k/F^{k-1}$$

is naturally isomorphic to the exterior algebra $\Lambda(V)$. Since the associated graded algebra of a filtered algebra is always isomorphic to the filtered algebra as filtered vector spaces (by choosing complements of F^k in F^{k+1} for all k), this provides an isomorphism (although not a natural one) in any characteristic, even two.

Grading

In the following, assume that the characteristic is not 2.[2]

Clifford algebras are $\mathbf{Z}_2$-graded algebras (also known as superalgebras). Indeed, the linear map on V defined by $v \mapsto -v$ (reflection through the origin) preserves the quadratic form Q and so by the universal property of Clifford algebras extends to an algebra automorphism

$$\alpha : C\ell(V,Q) \to C\ell(V,Q).$$

Since α is an involution (i.e. it squares to the identity) one can decompose $C\ell(V,Q)$ into positive and negative eigenspaces

$$C\ell(V,Q) = C\ell^0(V,Q) \oplus C\ell^1(V,Q)$$

where $C\ell^i(V,Q) = \{x \in C\ell(V,Q) \mid \alpha(x) = (-1)^i x\}$. Since α is an automorphism it follows that

$$C\ell^i(V,Q)C\ell^j(V,Q) = C\ell^{i+j}(V,Q)$$

where the superscripts are read modulo 2. This gives $C\ell(V,Q)$ the structure of a $\mathbf{Z}_2$-graded algebra. The subspace $C\ell^0(V,Q)$ forms a subalgebra of $C\ell(V,Q)$, called the *even subalgebra*. The subspace $C\ell^1(V,Q)$ is called the *odd part* of $C\ell(V,Q)$ (it is not a subalgebra). The $\mathbf{Z}_2$-grading plays an important role in the analysis and application of Clifford algebras. The automorphism α is called the *main involution* or *grade involution*.

Remark. In characteristic not 2 the underlying vector space of $C\ell(V,Q)$ inherits a **Z**-grading from the canonical isomorphism with the underlying vector space of the exterior algebra $\Lambda(V)$. It is important to note, however, that this is a *vector space grading only*. That is, Clifford multiplication does not respect the **Z**-grading, only the $\mathbf{Z}_2$-grading: for instance if $Q(v) \neq 0$, then $v \in C\ell^1(V,Q)$, but $v^2 \in C\ell^0(V,Q)$, not in $C\ell^2(V,Q)$. Happily, the gradings are related in the natural way: $\mathbf{Z}_2 = \mathbf{Z}/2\mathbf{Z}$. Further, the Clifford algebra is **Z**-filtered: $C\ell^{\leq i}(V,Q) \cdot C\ell^{\leq j}(V,Q) \subset C\ell^{\leq i+j}(V,Q)$. The *degree* of a Clifford number usually refers to the degree in the **Z**-grading. Elements which are pure in the $\mathbf{Z}_2$-grading are simply said to be even or odd.

The even subalgebra $C\ell^0(V,Q)$ of a Clifford algebra is itself a Clifford algebra[3] . If V is the orthogonal direct sum of a vector a of norm $Q(a)$ and a subspace U, then $C\ell^0(V,Q)$ is isomorphic to $C\ell(U,-Q(a)Q)$, where $-Q(a)Q$ is the form Q restricted to U and multiplied by $-Q(a)$. In particular over the reals this implies that

$C\ell^0_{p,q}(\mathbb{R}) \cong C\ell_{p,q-1}(\mathbb{R})$ for $q > 0$, and

$C\ell^0_{p,q}(\mathbb{R}) \cong C\ell_{q,p-1}(\mathbb{R})$ for $p > 0$.

In the negative-definite case this gives an inclusion $C\ell_{0,n-1}(\mathbf{R}) \subset C\ell_{0,n}(\mathbf{R})$ which extends the sequence

$\mathbf{R} \subset \mathbf{C} \subset \mathbf{H} \subset \mathbf{H}\oplus\mathbf{H} \subset \ldots$

Likewise, in the complex case, one can show that the even subalgebra of $C\ell_n(\mathbf{C})$ is isomorphic to $C\ell_{n-1}(\mathbf{C})$.

Antiautomorphisms

In addition to the automorphism α, there are two antiautomorphisms which play an important role in the analysis of Clifford algebras. Recall that the tensor algebra $T(V)$ comes with an antiautomorphism that reverses the order in all products:

$$v_1 \otimes v_2 \otimes \cdots \otimes v_k \mapsto v_k \otimes \cdots \otimes v_2 \otimes v_1.$$

Since the ideal I_Q is invariant under this reversal, this operation descends to an antiautomorphism of $C\ell(V,Q)$ called the *transpose* or *reversal* operation, denoted by x^t. The transpose is an antiautomorphism: $(xy)^t = y^t x^t$. The transpose operation makes no use of the $\mathbf{Z}_2$-grading so we define a second antiautomorphism by composing α and the transpose. We call this operation *Clifford conjugation* denoted $\bar{x}$

$$\bar{x} = \alpha(x^t) = \alpha(x)^t.$$

Of the two antiautomorphisms, the transpose is the more fundamental.[4]

Note that all of these operations are involutions. One can show that they act as ±1 on elements which are pure in the **Z**-grading. In fact, all three operations depend only on the degree modulo 4. That is, if x is pure with degree k then

$$\alpha(x) = \pm x \qquad x^t = \pm x \qquad \bar{x} = \pm x$$

where the signs are given by the following table:

***k* mod 4**	**0**	**1**	**2**	**3**	
$\alpha(x)$	+	−	+	−	$(-1)^k$
x^t	+	+	−	−	$(-1)^{k(k-1)/2}$
$\bar{x}$	+	−	−	+	$(-1)^{k(k+1)/2}$

The Clifford scalar product

When the characteristic is not 2 the quadratic form Q on V can be extended to a quadratic form on all of $C\ell(V,Q)$ as explained earlier (which we also denoted by Q). A basis independent definition is

$$Q(x) = \langle x^t x \rangle$$

where $\langle a \rangle$ denotes the scalar part of a (the grade 0 part in the **Z**-grading). One can show that

$$Q(v_1 v_2 \cdots v_k) = Q(v_1) Q(v_2) \cdots Q(v_k)$$

where the v_i are elements of V — this identity is *not* true for arbitrary elements of $C\ell(V,Q)$.

The associated symmetric bilinear form on $C\ell(V,Q)$ is given by

$$\langle x, y \rangle = \langle x^t y \rangle.$$

One can check that this reduces to the original bilinear form when restricted to V. The bilinear form on all of $C\ell(V,Q)$ is nondegenerate if and only if it is nondegenerate on V.

It is not hard to verify that the transpose is the adjoint of left/right Clifford multiplication with respect to this inner product. That is,

$$\langle ax, y \rangle = \langle x, a^t y \rangle, \text{and}$$
$$\langle xa, y \rangle = \langle x, y a^t \rangle.$$

Structure of Clifford algebras

In this section we assume that the vector space V is finite dimensional and that the bilinear form of Q is non-singular. A central simple algebra over K is a matrix algebra over a (finite dimensional) division algebra with center K. For example, the central simple algebras over the reals are matrix algebras over either the reals or the quaternions.

- If V has even dimension then $C\ell(V,Q)$ is a central simple algebra over K.
- If V has even dimension then $C\ell^0(V,Q)$ is a central simple algebra over a quadratic extension of K or a sum of two isomorphic central simple algebras over K.
- If V has odd dimension then $C\ell(V,Q)$ is a central simple algebra over a quadratic extension of K or a sum of two isomorphic central simple algebras over K.
- If V has odd dimension then $C\ell^0(V,Q)$ is a central simple algebra over K.

The structure of Clifford algebras can be worked out explicitly using the following result. Suppose that U has even dimension and a non-singular bilinear form with discriminant d, and suppose that V is another vector space with a quadratic form. The Clifford algebra of $U+V$ is isomorphic to the tensor product of the Clifford algebras of U and $(-1)^{\dim(U)/2}dV$, which is the space V with its quadratic form multiplied by $(-1)^{\dim(U)/2}d$. Over the reals, this implies in particular that

$$Cl_{p+2,q}(\mathbb{R}) = M_2(\mathbb{R}) \otimes Cl_{q,p}(\mathbb{R})$$
$$Cl_{p+1,q+1}(\mathbb{R}) = M_2(\mathbb{R}) \otimes Cl_{p,q}(\mathbb{R})$$
$$Cl_{p,q+2}(\mathbb{R}) = \mathbb{H} \otimes Cl_{q,p}(\mathbb{R}).$$

These formulas can be used to find the structure of all real Clifford algebras; see the classification of Clifford algebras.

Notably, the Morita equivalence class of a Clifford algebra (its representation theory: the equivalence class of the category of modules over it) depends only on the signature $p - q$ mod 8. This is an algebraic form of Bott periodicity.

The Clifford group Γ

In this section we assume that V is finite dimensional and the quadratic form Q is nondegenerate.

The invertible elements of the Clifford algebra act on it by twisted conjugation: conjugation by x maps $y \mapsto xy\alpha(x)^{-1}$.

The Clifford group Γ is defined to be the set of invertible elements x that *stabilize vectors*, meaning that

$$xv\alpha(x)^{-1} \in V$$

for all v in V.

This formula also defines an action of the Clifford group on the vector space V that preserves the norm Q, and so gives a homomorphism from the Clifford group to the orthogonal group. The Clifford group contains all elements r of V of nonzero norm, and these act on V by the corresponding reflections that take v to $v - \langle v,r\rangle r/Q(r)$ (In characteristic 2 these are called orthogonal transvections rather than reflections.)

The Clifford group Γ is the disjoint union of two subsets Γ^0 and Γ^1, where Γ^i is the subset of elements of degree i. The subset Γ^0 is a subgroup of index 2 in Γ.

If V is a finite dimensional real vector space with positive definite (or negative definite) quadratic form then the Clifford group maps onto the orthogonal group of V with respect to the form (by the Cartan-Dieudonné theorem) and the kernel consists of the nonzero elements of the field K. This leads to exact sequences

$$1 \to K^* \to \Gamma \to O_V(K) \to 1,$$
$$1 \to K^* \to \Gamma^0 \to SO_V(K) \to 1.$$

Over other fields or with indefinite forms, the map is not in general onto, and the failure is captured by the spinor norm.

Spinor norm

In arbitrary characteristic, the spinor norm Q is defined on the Clifford group by

$$Q(x) = x^t x.$$

It is a homomorphism from the Clifford group to the group K^* of non-zero elements of K. It coincides with the quadratic form Q of V when V is identified with a subspace of the Clifford algebra. Several authors define the spinor norm slightly differently, so that it differs from the one here by a factor of −1, 2, or −2 on Γ^1. The difference is not very important in characteristic other than 2.

The nonzero elements of K have spinor norm in the group K^{*2} of squares of nonzero elements of the field K. So when V is finite dimensional and non-singular we get an induced map from the orthogonal group of V to the group K^*/K^{*2}, also called the spinor norm. The spinor norm of the reflection of a vector r has image $Q(r)$ in K^*/K^{*2}, and this property uniquely defines it on the orthogonal group. This gives exact sequences:

$$1 \to \{\pm 1\} \to \mathrm{Pin}_V(K) \to O_V(K) \to K^*/K^{*2},$$
$$1 \to \{\pm 1\} \to \mathrm{Spin}_V(K) \to SO_V(K) \to K^*/K^{*2}.$$

Note that in characteristic 2 the group {±1} has just one element.

From the point of view of Galois cohomology of algebraic groups, the spinor norm is a connecting homomorphism on cohomology. Writing μ_2 for the algebraic group of square roots of 1 (over a field of characteristic not 2 it is roughly the same as a two-element group with trivial Galois action), the short exact sequence

$$1 \to \mu_2 \to \mathrm{Pin}_V \to O_V \to 1$$

yields a long exact sequence on cohomology, which begins

$$1 \to H^0(\mu_2; K) \to H^0(\mathrm{Pin}_V; K) \to H^0(O_V; K) \to H^1(\mu_2; K).$$

The 0th Galois cohomology group of an algebraic group with coefficients in K is just the group of K-valued points: $H^0(G;K) = G(K)$, and $H^1(\mu_2;K) \cong K^*/K^{*2}$, which recovers the previous sequence

$$1 \to \{\pm 1\} \to \mathrm{Pin}_V(K) \to O_V(K) \to K^*/K^{*2},$$

where the spinor norm is the connecting homomorphism $H^0(O_V;K) \to H^1(\mu_2;K)$.

Spin and Pin groups

In this section we assume that V is finite dimensional and its bilinear form is non-singular. (If K has characteristic 2 this implies that the dimension of V is even.)

The Pin group $Pin_V(K)$ is the subgroup of the Clifford group Γ of elements of spinor norm 1, and similarly the Spin group $Spin_V(K)$ is the subgroup of elements of Dickson invariant 0 in $Pin_V(K)$. When the characteristic is not 2, these are the elements of determinant 1. The Spin group usually has index 2 in the Pin group.

Recall from the previous section that there is a homomorphism from the Clifford group onto the orthogonal group. We define the special orthogonal group to be the image of Γ^0. If K does not have characteristic 2 this is just the group of elements of the orthogonal group of determinant 1. If K does have characteristic 2, then all elements of the orthogonal group have determinant 1, and the special orthogonal group is the set of elements of Dickson invariant 0.

There is a homomorphism from the Pin group to the orthogonal group. The image consists of the elements of spinor norm $1 \in K^*/K^{*2}$. The kernel consists of the elements +1 and −1, and has order 2 unless K has characteristic 2. Similarly there is a homomorphism from the Spin group to the special orthogonal group of V.

In the common case when V is a positive or negative definite space over the reals, the spin group maps onto the special orthogonal group, and is simply connected when V has dimension at least 3. Further the kernel of this homomorphism consists of 1 and -1.So in this case the spin group, Spin(n), is a double cover of SO(n). Please note, however, that the simple connectedness of the spin group is not true in general: if V is $R^{p,q}$ for p and q both at least 2 then the spin group is not simply connected. In this case the algebraic group $Spin_{p,q}$ is simply connected as an algebraic group, even though its group of real valued points $Spin_{p,q}(R)$ is not simply connected. This is a rather subtle point, which completely confused the authors of at least one standard book about spin groups.

Spinors

Clifford algebras $C\ell_{p,q}(\mathbf{C})$, with $p+q=2n$ even, are matrix algebras which have a complex representation of dimension 2^n. By restricting to the group $Pin_{p,q}(\mathbf{R})$ we get a complex representation of the Pin group of the same dimension, called the spin representation. If we restrict this to the spin group $Spin_{p,q}(\mathbf{R})$ then it splits as the sum of two *half spin representations* (or *Weyl representations*) of dimension 2^{n-1}.

If $p+q=2n+1$ is odd then the Clifford algebra $C\ell_{p,q}(\mathbf{C})$ is a sum of two matrix algebras, each of which has a representation of dimension 2^n, and these are also both representations of the Pin group $Pin_{p,q}(\mathbf{R})$. On restriction to the spin group $Spin_{p,q}(\mathbf{R})$ these become isomorphic, so the spin group has a complex spinor representation of dimension 2^n.

More generally, spinor groups and pin groups over any field have similar representations whose exact structure depends on the structure of the corresponding Clifford algebras: whenever a Clifford algebra has a factor that is a matrix algebra over some division algebra, we get a corresponding representation of the pin and spin groups over that division algebra. For examples over the reals see the article on spinors.

Real spinors

To describe the real spin representations, one must know how the spin group sits inside its Clifford algebra. The Pin group, $Pin_{p,q}$ is the set of invertible elements in $Cl_{p,q}$ which can be written as a product of unit vectors:

$$Pin_{p,q} = \{v_1 v_2 \dots v_r \mid \forall i, \|v_i\| = \pm 1\}.$$

Comparing with the above concrete realizations of the Clifford algebras, the Pin group corresponds to the products of arbitrarily many reflections: it is a cover of the full orthogonal group $O(p,q)$. The Spin group consists of those elements of $Pin_{p,q}$ which are products of an even number of unit vectors. Thus by the Cartan-Dieudonné theorem *Spin* is a cover of the group of proper rotations $SO(p,q)$.

Let $\alpha : C\ell \to C\ell$ be the automorphism which is given by $-Id$ acting on pure vectors. Then in particular, $Spin_{p,q}$ is the subgroup of $Pin_{p,q}$ whose elements are fixed by α. Let

$$Cl^0_{p,q} = \{x \in Cl_{p,q} \mid \alpha(x) = x\}.$$

(These are precisely the elements of even degree in $C\ell_{p,q}$.) Then the spin group lies within $C\ell^0_{p,q}$.

The irreducible representations of $C\ell_{p,q}$ restrict to give representations of the pin group. Conversely, since the pin group is generated by unit vectors, all of its irreducible representation are induced in this manner. Thus the two representations coincide. For the same reasons, the irreducible representations of the spin coincide with the irreducible representations of $C\ell^0_{p,q}$

To classify the pin representations, one need only appeal to the classification of Clifford algebras. To find the spin representations (which are representations of the even subalgebra), one can first make use of either of the isomorphisms (see above)

$C\ell^0_{p,q} \approx C\ell_{p,q-1}$, for $q > 0$

$C\ell^0_{p,q} \approx C\ell_{q,p-1}$, for $p > 0$

and realize a spin representation in signature (p,q) as a pin representation in either signature $(p,q-1)$ or $(q,p-1)$.

Applications

Differential geometry

One of the principal applications of the exterior algebra is in differential geometry where it is used to define the bundle of differential forms on a smooth manifold. In the case of a (pseudo-)Riemannian manifold, the tangent spaces come equipped with a natural quadratic form induced by the metric. Thus, one can define a Clifford bundle in analogy with the exterior bundle. This has a number of important applications in Riemannian geometry. Perhaps more importantly is the link to a spin manifold, its associated spinor bundle and spin^c manifolds.

Physics

Clifford algebras have numerous important applications in physics. Physicists usually consider a Clifford algebra to be an algebra spanned by matrices $\gamma_0,\dots,\gamma_3$ called Dirac matrices which have the property that

$$\gamma_i\gamma_j + \gamma_j\gamma_i = 2\eta_{ij}$$

where η is the matrix of a quadratic form of signature (1,3). These are exactly the defining relations for the Clifford algebra $Cl_{1,3}(C)$ (up to an unimportant factor of 2), which by the classification of Clifford algebras is isomorphic to the algebra of 4 by 4 complex matrices.

The Dirac matrices were first written down by Paul Dirac when he was trying to write a relativistic first-order wave equation for the electron, and give an explicit isomorphism from the Clifford algebra to the algebra of complex matrices. The result was used to define the Dirac equation and introduce the Dirac operator. The entire Clifford algebra shows up in quantum field theory in the form of Dirac field bilinears.

Computer Vision

Recently, Clifford algebras have been applied in the problem of action recognition and classification in computer vision. Rodriguez et al. [5] propose a Clifford embedding to generalize traditional MACH filters to video (3D spatiotemporal volume), and vector-valued data such as optical flow. Vector-valued data is analyzed using the Clifford Fourier transform. Based on these vectors action filters are synthesized in the Clifford Fourier domain and recognition of actions is performed using Clifford Correlation. The authors demonstrate the effectiveness of the Clifford embedding by recognizing actions typically performed in classic feature films and sports broadcast television.

Notes

[1] Mathematicians who work with real Clifford algebras and prefer positive definite quadratic forms (especially those working in index theory) sometimes use a different choice of sign in the fundamental Clifford identity. That is, they take $v^2 = -Q(v)$. One must replace Q with $-Q$ in going from one convention to the other.

[2] Thus the group algebra **K**[**Z**/2] is semisimple and the Clifford algebra splits into eigenspaces of the main involution.

[3] We are still assuming that the characteristic is not 2.

[4] The opposite is true when using the alternate (−) sign convention for Clifford algebras: it is the conjugate which is more important. In general, the meanings of conjugation and transpose are interchanged when passing from one sign convention to the other. For example, in the convention used here the inverse of a vector is given by $v^{-1} = v^t/Q(v)$ while in the (−) convention it is given by $v^{-1} = \bar{v}/Q(v)$.

[5] Rodriguez, Mikel; Shah, M (2008). "Action MACH: A Spatio-Temporal Maximum Average Correlation Height Filter for Action Classification". *Computer Vision and Pattern Recognition (CVPR).*

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External links

- Planetmath entry on Clifford algebras (http://planetmath.org/encyclopedia/CliffordAlgebra2.html)
- A history of Clifford algebras (http://members.fortunecity.com/jonhays/clifhistory.htm) (unverified)
- John Baez on Clifford algebras (http://www.math.ucr.edu/home/baez/octonions/node6.html)

Von Neumann algebra

In mathematics, a **von Neumann algebra** or **W*-algebra** is a *-algebra of bounded operators on a Hilbert space that is closed in the weak operator topology and contains the identity operator. They were originally introduced by John von Neumann, motivated by the study of single operators, group representations, ergodic theory and quantum mechanics. His double commutant theorem shows that the analytic definition is equivalent to a purely algebraic definition as an algebra of symmetries.

Two basic examples of von Neumann algebras are as follows. The ring $L^\infty(\mathbf{R})$ of essentially bounded measurable functions on the real line is a commutative von Neumann algebra, which acts by pointwise multiplication on the Hilbert space $L^2(\mathbf{R})$ of square integrable functions. The algebra $B(H)$ of all bounded operators on a Hilbert space H is a von Neumann algebra, non-commutative if the Hilbert space has dimension at least 2.

Von Neumann algebras were first studied by von Neumann (1929); he and Francis Murray developed the basic theory, under the original name of **rings of operators**, in a series of papers written in the 1930s and 1940s (F.J. Murray & J. von Neumann 1936, 1937, 1943; J. von Neumann 1938, 1940, 1943, 1949), reprinted in the collected works of von Neumann (1961).

Introductory accounts of von Neumann algebras are given in the online notes of Jones (2003) and Wassermann (1991) and the books by Dixmier (1981), Schwartz (1967), Blackadar (2005) and Sakai (1971). The three volume work by Takesaki (1979) gives an encyclopedic account of the theory. The book by Connes (1994) discusses more advanced topics.

Definitions

There are three common ways to define von Neumann algebras.

The first and most common way is to define them as weakly closed * algebras of bounded operators (on a Hilbert space) containing the identity. In this definition the weak (operator) topology can be replaced by many other common topologies including the strong, ultrastrong or ultraweak operator topologies. The *-algebras of bounded operators that are closed in the norm topology are C*-algebras, so in particular any von Neumann algebra is a C*-algebra.

The second definition is that a von Neumann algebra is a subset of the bounded operators closed under * and equal to its double commutant, or equivalently the commutant of some subset closed under *. The von Neumann double commutant theorem (von Neumann 1929) says that the first two definitions are equivalent.

The first two definitions describe a von Neumann algebras concretely as a set of operators acting on some given Hilbert space. Sakai (1971) showed that von Neumann algebras can also be defined abstractly as C*-algebras that have a predual; in other words the von Neumann algebra, considered as a Banach space, is the dual of some other Banach space called the predual. The predual of a von Neumann algebra is in fact unique up to isomorphism. Some authors use "von Neumann algebra" for the algebras together with a Hilbert space action, and "W*-algebra" for the abstract concept, so a von Neumann algebra is a W*-algebra together with a Hilbert space and a suitable faithful unital action on the Hilbert space. The concrete and abstract definitions of a von Neumann algebra are similar to the concrete and abstract definitions of a C*-algebra, which can be defined either as norm-closed * algebras of operators on a Hilbert space, or as Banach *-algebras such that $\|a\, a^*\|=\|a\|\, \|a^*\|$.

Terminology

Some of the terminology in von Neumann algebra theory can be confusing, and the terms often have different meanings outside the subject.

- A **factor** is a von Neumann algebra with trivial center, i.e. a center consisting only of scalar operators.
- A **finite** von Neumann algebra is one which is the direct integral of finite factors. Similarly, **properly infinite** von Neumann algebras are the direct integral of properly infinite factors.
- A von Neumann algebra that acts on a separable Hilbert space is called **separable**. Note that such algebras are rarely separable in the norm topology.
- The von Neumann algebra **generated** by a set of bounded operators on a Hilbert space is the smallest von Neumann algebra containing all those operators.
- The **tensor product** of two von Neumann algebras acting on two Hilbert spaces is defined to be the von Neumann algebra generated by their algebraic tensor product, considered as operators on the Hilbert space tensor product of the Hilbert spaces.

By forgetting about the topology on a von Neumann algebra, we can consider it a (unital) *-algebra, or just a ring. Von Neumann algebras are semihereditary: every finitely generated submodule of a projective module is itself projective. There have been several attempts to axiomatize the underlying rings of von Neumann algebras, including Baer *-rings and AW* algebras. The *-algebra of affiliated operators of a finite von Neumann algebra is a von Neumann regular ring. (The von Neumann algebra itself is in general not von Neumann regular.)

Commutative von Neumann algebras

Main article: Abelian von Neumann algebra

The relationship between commutative von Neumann algebras and measure spaces is analogous to that between commutative C*-algebras and locally compact Hausdorff spaces. Every commutative von Neumann algebra is isomorphic to $L^\infty(X)$ for some measure space (X, μ) and conversely, for every σ-finite measure space X, the * algebra $L^\infty(X)$ is a von Neumann algebra.

Due to this analogy, the theory of von Neumann algebras has been called noncommutative measure theory, while the theory of C*-algebras is sometimes called noncommutative topology (Connes 1994).

Projections

Operators E in a von Neumann algebra for which $E = EE = E^*$ are called **projections**; they are exactly the operators which give an orthogonal projection of H onto some closed subspace. A subspace of the Hilbert space H is said to **belong to** the von Neumann algebra M if it is the image of some projection in M. Informally these are the closed subspaces that can be described using elements of M, or that M "knows" about. The closure of the image of any operator in M, or the kernel of any operator in M belong to M, and the closure of the image of any subspace belonging to M under an operator of M also belongs to M. There is a 1:1 correspondence between projections of M and subspaces that belong to it.

The basic theory of projections was worked out by Murray & von Neumann (1936). Two subspaces belonging to M are called (**Murray-von Neumann**) **equivalent** if there is a partial isometry mapping the first isomorphically onto the other that is an element of the von Neumann algebra (informally, if M "knows" that the subspaces are isomorphic). This induces a natural equivalence relation on projections by defining E to be equivalent to F if the corresponding subspaces are equivalent, or in other words if there is a partial isometry of H that maps the image of E isometrically to the image of F and is an element of the von Neumann algebra. Another way of stating this is that E is equivalent to F if $E=uu^*$ and $F=u^*u$ for some partial isometry u in M.

The equivalence relation ~ thus defined is additive in the following sense: Suppose $E_1 \sim F_1$ and $E_2 \sim F_2$. If $E_1 \perp E_2$ and $F_1 \perp F_2$, then $E_1 + E_2 \sim F_1 + F_2$. This is not true in general if one requires unitary equivalence in the definition of

~, i.e. if we say E is equivalent to F if $u^*Eu = F$ for some unitary u. .

The subspaces belonging to M are partially ordered by inclusion, and this induces a partial order ≤ of projections. There is also a natural partial order on the set of *equivalence classes* of projections, induced by the partial order ≤ of projections. If M is a factor, ≤ is a total order on equivalence classes of projections, described in the section on traces below.

A projection (or subspace belonging to M) E is said to be *finite* if there is no projection $F < E$ that is equivalent to E. For example, all finite-dimensional projections (or subspaces) are finite (since isometries between Hilbert spaces leave the dimension fixed), but the identity operator on an infinite-dimensional Hilbert space is not finite in the von Neumann algebra of all bounded operators on it, since it is isometrically isomorphic to a proper subset of itself. However it is possible for infinite dimensional subspaces to be finite.

Orthogonal projections are noncommutative analogues of indicator functions in $L^\infty(\mathbf{R})$. $L^\infty(\mathbf{R})$ is the $\|\cdot\|_\infty$-closure of the subspace generated by the indicator functions. Similarly, a von Neumann algebra is generated by its projections; this is a consequence of the spectral theorem for self-adjoint operators.

Factors

A von Neumann algebra N whose center consists only of multiples of the identity operator is called a **factor**. von Neumann (1949) showed that every von Neumann algebra on a separable Hilbert space is isomorphic to a direct integral of factors. This decomposition is essentially unique. Thus, the problem of classifying isomorphism classes of von Neumann algebras on separable Hilbert spaces can be reduced to that of classifying isomorphism classes of factors.

Murray & von Neumann (1936) showed that every factor has one of 3 types as described below. The type classification can be extended to von Neumann algebras that are not factors, and a von Neumann algebra is of type X if it can be decomposed as a direct integral of type X factors; for example, every commutative von Neumann algebra has type I_1. Every von Neumann algebra can be written uniquely as a sum of von Neumann algebras of types I, II, and III.

There are several other ways to divide factors into classes that are sometimes used:

- A factor is called **discrete** (or occasionally **tame**) if it has type I, and **continuous** (or occasionally **wild**) if it has type II or III.
- A factor is called **semifinite** if it has type I or II, and **purely infinite** if it has type III.
- A factor is called **finite** if the projection 1 is finite and **properly infinite** otherwise. Factors of types I and II may be either finite or properly infinite, but factors of type III are always properly infinite.

Type I factors

A factor is said to be of **type I** if there is a minimal projection $E \neq 0$, i.e. a projection E such that there is no other projection F with 0 < F < E. Any factor of type I is isomorphic to the von Neumann algebra of *all* bounded operators on some Hilbert space; since there is one Hilbert space for every cardinal number, isomorphism classes of factors of type I correspond exactly to the cardinal numbers. Since many authors consider von Neumann algebras only on separable Hilbert spaces, it is customary to call the bounded operators on a Hilbert space of finite dimension n a factor of type I_n, and the bounded operators on a separable infinite-dimensional Hilbert space, a factor of type I_∞.

Type II factors

A factor is said to be of **type II** if there are no minimal projections but there are non-zero finite projections. This implies that every projection *E* can be halved in the sense that there are equivalent projections *F* and *G* such that *E* = *F* + *G*. If the identity operator in a type II factor is finite, the factor is said to be of type II_1; otherwise, it is said to be of type II_∞. The best understood factors of type II are the hyperfinite type II_1 factor and the hyperfinite type II_∞ factor, found by Murray & von Neumann (1936). These are the unique hyperfinite factors of types II_1 and II_∞; there are an uncountable number of other factors of these types that are the subject of intensive study. Murray & von Neumann (1937) proved the fundamental result that a factor of type II_1 has a unique finite tracial state, and the set of traces of projections is [0,1].

A factor of type II_∞ has a semifinite trace, unique up to rescaling, and the set of traces of projections is $[0,\infty]$. The set of real numbers λ such that there is an automorphism rescaling the trace by a factor of λ is called the **fundamental group** of the type II_∞ factor.

The tensor product of a factor of type II_1 and an infinite type I factor has type II_∞, and conversely any factor of type II_∞ can be constructed like this. The **fundamental group** of a type II_1 factor is defined to be the fundamental group of its tensor product with the infinite (separable) factor of type I. For many years it was an open problem to find a type II factor whose fundamental group was not the group of all positive reals, but Connes then showed that the von Neumann group algebra of a countable discrete group with Kazhdan's property T (the trivial representation is isolated in the dual space), such as $SL_3(\mathbf{Z})$, has a countable fundamental group. Subsequently Sorin Popa showed that the fundamental group can be trivial for certain groups, including the semidirect product of $\mathbf{Z}^2$ by $SL_2(\mathbf{Z})$.

An example of a type II_1 factor is the von Neumann group algebra of a countable infinite discrete group such that every non-trivial conjugacy class is infinite. McDuff (1969) found an uncountable family of such groups with non-isomorphic von Neumann group algebras, thus showing the existence of uncountably many different separable type II_1 factors.

Type III factors

Lastly, **type III** factors are factors that do not contain any nonzero finite projections at all. In their first paper Murray & von Neumann (1936) were unable to decide whether or not they existed; the first examples were later found by von Neumann (1940). Since the identity operator is always infinite in those factors, they were sometimes called type III_∞ in the past, but recently that notation has been superseded by the notation III_λ, where λ is a real number in the interval [0,1]. More precisely, if the Connes spectrum (of its modular group) is 1 then the factor is of type III_0, if the Connes spectrum is all integral powers of λ for $0 < \lambda < 1$, then the type is III_λ, and if the Connes spectrum is all positive reals then the type is III_1. (The Connes spectrum is a closed subgroup of the positive reals, so these are the only possibilities.) The only trace on type III factors takes value ∞ on all non-zero positive elements, and any two non-zero projections are equivalent. At one time type III factors were considered to be intractable objects, but Tomita–Takesaki theory has led to a good structure theory. In particular, any type III factor can be written in a canonical way as the crossed product of a type II_∞ factor and the real numbers.

The predual

Any von Neumann algebra M has a **predual** M_*, which is the Banach space of all ultraweakly continuous linear functionals on M. As the name suggests, M is (as a Banach space) the dual of its predual. The predual is unique in the sense that any other Banach space whose dual is M is canonically isomorphic to M_*. Sakai (1971) showed that the existence of a predual characterizes von Neumann algebras among C* algebras.

The definition of the predual given above seems to depend on the choice of Hilbert space that M acts on, as this determines the ultraweak topology. However the predual can also be defined without using the Hilbert space that M acts on, by defining it to be the space generated by all positive **normal** linear functionals on M. (Here "normal" means that it preserves suprema when applied to increasing nets of self adjoint operators; or equivalently to increasing sequences of projections.)

The predual M_* is a closed subspace of the dual M^* (which consists of all norm-continuous linear functionals on M) but is generally smaller. The proof that M_* is (usually) not the same as M^* is nonconstructive and uses the axiom of choice in an essential way; it is very hard to exhibit explicit elements of M^* that are not in M_*. For example, exotic positive linear forms on the von Neumann algebra $l^\infty(Z)$ are given by free ultrafilters; they correspond to exotic *-homomorphisms into C and describe the Stone-Cech compactification of Z.

Examples:

1. The predual of the von Neumann algebra $L^\infty(\mathbf{R})$ of essentially bounded functions on $\mathbf{R}$ is the Banach space $L^1(\mathbf{R})$ of integrable functions. The dual of $L^\infty(\mathbf{R})$ is strictly larger than $L^1(\mathbf{R})$ For example, a functional on $L^\infty(\mathbf{R})$ that extends the Dirac measure δ_0 on the closed subspace of bounded continuous functions $C^0{}_b(\mathbf{R})$ cannot be represented as a function in $L^1(\mathbf{R})$.
2. The predual of the von Neumann algebra $B(H)$ of bounded operators on a Hilbert space H is the Banach space of all trace class operators with the trace norm $\|A\|= \mathrm{Tr}(|A|)$. The Banach space of trace class operators is itself the dual of the C*-algebra of compact operators (which is not a von Neumann algebra).

Weights, states, and traces

Weights and their special cases states and traces are discussed in detail in (Takesaki 1979).

- A **weight** ω on a von Neumann algebra is a linear map from the set of positive elements (those of the form a^*a) to $[0,\infty]$.
- A **positive linear functional** is a weight with ω(1) finite (or rather the extension of ω to the whole algebra by linearity).
- A **state** is a weight with ω(1)=1.
- A **trace** is a weight with $\omega(aa^*)=\omega(a^*a)$ for all a.
- A **tracial state** is a trace with ω(1)=1.

Any factor has a trace such that the trace of a non-zero projection is non-zero and the trace of a projection is infinite if and only if the projection is infinite. Such a trace is unique up to rescaling. For factors that are separable or finite, two projections are equivalent if and only if they have the same trace. The type of a factor can be read off from the possible values of this trace as follows:

- Type I_n: 0, x, $2x$, ...,nx for some positive x (usually normalized to be $1/n$ or 1).
- Type I_∞: 0, x, $2x$, ...,∞ for some positive x (usually normalized to be 1).
- Type II_1: $[0,x]$ for some positive x (usually normalized to be 1).
- Type II_∞: $[0,\infty]$.
- Type III: 0,∞.

If a von Neumann algebra acts on a Hilbert space containing a norm 1 vector v, then the functional $a \to (av,v)$ is a normal state. This construction can be reversed to give an action on a Hilbert space from a normal state: this is the

GNS construction for normal states.

Modules over a factor

Given an abstract separable factor, one can ask for a classification of its modules, meaning the separable Hilbert spaces that it acts on. The answer is given as follows: every such module *H* can be given an *M*-dimension $\dim_M(H)$ (not its dimension as a complex vector space) such that modules are isomorphic if and only if they have the same *M*-dimension. The *M*-dimension is additive, and a module is isomorphic to a subspace of another module if and only if it has smaller or equal *M*-dimension.

A module is called **standard** if it has a cyclic separating vector. Each factor has a standard representation, which is unique up to isomorphism. The standard representation has an antilinear involution *J* such that $JMJ = M'$. For finite factors the standard module is given by the GNS construction applied to the unique normal tracial state and the *M*-dimension is normalized so that the standard module has *M*-dimension 1, while for infinite factors the standard module is the module with *M*-dimension equal to ∞.

The possible *M*-dimensions of modules are given as follows:

- Type I_n (*n* finite): The *M*-dimension can be any of 0/*n*, 1/*n*, 2/*n*, 3/*n*, ..., ∞. The standard module has *M*-dimension 1 (and complex dimension n^2.)
- Type I_∞ The *M*-dimension can be any of 0, 1, 2, 3, ..., ∞. The standard representation of *B*(*H*) is $H\otimes H$; its *M*-dimension is ∞.
- Type II_1: The *M*-dimension can be anything in [0, ∞]. It is normalized so that the standard module has *M*-dimension 1. The *M*-dimension is also called the **coupling constant** of the module *H*.
- Type II_∞: The *M*-dimension can be anything in [0, ∞]. There is in general no canonical way to normalize it; the factor may have outer automorphisms multiplying the *M*-dimension by constants. The standard representation is the one with *M*-dimension ∞.
- Type III: The *M*-dimension can be 0 or ∞. Any two non-zero modules are isomorphic, and all non-zero modules are standard.

Amenable von Neumann algebras

Connes (1976) and others proved that the following conditions on a von Neumann algebra *M* on a separable Hilbert space *H* are all equivalent:

- *M* is **hyperfinite** or **AFD** or **approximately finite dimensional** or **approximately finite**: this means the algebra contains an ascending sequence of finite dimensional subalgebras with dense union. (Warning: some authors use "hyperfinite" to mean "AFD and finite".)
- *M* is **amenable**: this means that the derivations of *M* with values in a normal dual Banach bimodule are all inner.
- *M* has Schwartz's **property P**: for any bounded operator *T* on *H* the weak operator closed convex hull of the elements uTu^* contains an element commuting with *M*.
- *M* is **semidiscrete**: this means the identity map from *M* to *M* is a weak pointwise limit of completely positive maps of finite rank.
- *M* has **property E** or the **Hakeda-Tomiyama extension property**: this means that there is a projection of norm 1 from bounded operators on *H* to *M* '.
- *M* is **injective**: any completely positive linear map from any self adjoint closed subspace containing 1 of any unital C^*-algebra *A* to *M* can be extended to a completely positive map from *A* to *M*.

There is no generally accepted term for the class of algebras above; Connes has suggested that **amenable** should be the standard term.

The amenable factors have been classified: there is a unique one of each of the types I_n, I_∞, II_1, II_∞, III_λ, for 0<λ≤ 1, and the ones of type III_0 correspond to certain ergodic flows. (For type III_0 calling this a classification is a little misleading, as it is known that there is no easy way to classify the corresponding ergodic flows.) The ones of type I and II_1 were classified by Murray & von Neumann (1943), and the remaining ones were classified by Connes (1976), except for the type III_1 case which was completed by Haagerup.

All amenable factors can be constructed using the **group-measure space construction** of Murray and von Neumann for a single ergodic transformation. In fact they are precisely the factors arising as crossed products by free ergodic actions of *Z* or Z_n on abelian von Neumann algebras $L^\infty(X)$. Type I factors occur when the measure space *X* is atomic and the action transitive. When *X* is diffuse or non-atomic, it is equivalent to [0,1] as a measure space. Type II factors occur when *X* admits an equivalent finite (II_1) or infinite (II_∞) measure, invariant under Z. Type III factors occur in the remaining cases where there is no invariant measure, but only an invariant measure class: these factors are called **Krieger factors**.

Tensor products of von Neumann algebras

The Hilbert space tensor product of two Hilbert spaces is the completion of their algebraic tensor product. One can define a tensor product of von Neumann algebras (a completion of the algebraic tensor product of the algebras considered as rings), which is again a von Neumann algebra, and act on the tensor product of the corresponding Hilbert spaces. The tensor product of two finite algebras is finite, and the tensor product of an infinite algebra and a non-zero algebra is infinite. The type of the tensor product of two von Neumann algebras (I, II, or III) is the maximum of their types. The **commutation theorem for tensor products** states that

$$(M \otimes N)' = M' \otimes N'$$

(where M' denotes the commutant of *M*).

The tensor product of an infinite number of von Neumann algebras, if done naively, is usually a ridiculously large non-separable algebra. Instead von Neumann (1938) showed that one should choose a state on each of the von Neumann algebras, use this to define a state on the algebraic tensor product, which can be used to product a Hilbert space and a (reasonably small) von Neumann algebra. Araki & Woods (1968) studied the case where all the factors are finite matrix algebras; these factors are called **Araki-Woods** factors or **ITPFI factors** (ITPFI stands for "infinite tensor product of finite type I factors"). The type of the infinite tensor product can vary dramatically as the states are changed; for example, the infinite tensor product of an infinite number of type I_2 factors can have any type depending on the choice of states. In particular Powers (1967) found an uncountable family of non-isomorphic hyperfinite type III_λ factors for 0<λ<1, called **Powers factors**, by taking an infinite tensor product of type I_2 factors, each with the state given by : $x \mapsto \mathrm{Tr}\begin{pmatrix} \frac{1}{\lambda+1} & 0 \\ 0 & \frac{\lambda}{\lambda+1} \end{pmatrix} x.$

All hyperfinite von Neumann algebras not of type III_0 are isomorphic to Araki-Woods factors, but there are uncountably many of type III_0 that are not.

Bimodules and subfactors

A **bimodule** (or correspondence) is a Hilbert space *H* with module actions of two commuting von Neumann algebras. Bimodules have a much richer structure than that of modules. Any bimodule over two factors always gives a subfactor since one of the factors is always contained in the commutant of the other. There is also a subtle relative tensor product operation due to Connes on bimodules. The theory of subfactors, initiated by Vaughan Jones, reconciles these two seemingly different points of view.

Bimodules are also important for the von Neumann group algebra *M* of a discrete group Γ. Indeed if *V* is any unitary representation of Γ, then, regarding Γ as the diagonal subgroup of Γ x Γ, the corresponding induced representation on l^2 (Γ,V) is naturally a bimodule for two commuting copies of *M*. Important representation theoretic properties of Γ can be formulated entirely in terms of bimodules and therefore make sense for the von Neumann algebra itself. For example Connes and Jones gave a definition of an analogue of Kazhdan's Property T for von Neumann algebras in this way.

Non-amenable factors

Von Neumann algebras of type I are always amenable, but for the other types there are an uncountable number of different non-amenable factors, which seem very hard to classify, or even distinguish from each other. Nevertheless Voiculescu has shown that the class of non-amenable factors coming from the group-measure space construction is **disjoint** from the class coming from group von Neumann algebras of free groups. Later Narutaka Ozawa proved that group von Neumann algebras of hyperbolic groups yield prime type II_1 factors, i.e. ones that cannot be factored as tensor products of type II_1 factors, a result first proved by Leeming Ge for free group factors using Voiculescu's free entropy. Popa's work on fundamental groups of non-amenable factors represents another significant advance. The theory of factors "beyond the hyperfinite" is rapidly expanding at present, with many new and surprising results; it has close links with rigidity phenomena in geometric group theory and ergodic theory.

Examples

- The essentially bounded functions on a σ-finite measure space form a commutative (type I_1) von Neumann algebra acting on the L^2 functions. For certain non-σ-finite measure spaces, usually considered pathological, $L^\infty(X)$ is not a von Neumann algebra; for example, the σ-algebra of measurable sets might be the countable-cocountable algebra on an uncountable set.
- The bounded operators on any Hilbert space form a von Neumann algebra, indeed a factor, of type I.
- If we have any unitary representation of a group G on a Hilbert space H then the bounded operators commuting with G form a von Neumann algebra G', whose projections correspond exactly to the closed subspaces of H invariant under G. Equivalent subrepresentations correspond to equivalent projections in G'. The double commutant G'' of G is also a von Neumann algebra.
- The **von Neumann group algebra** of a discrete group G is the algebra of all bounded operators on $H = l^2(G)$ commuting with the action of G on H through right multiplication. One can show that this is the von Neumann algebra generated by the operators corresponding to multiplication from the left with an element $g \in G$. It is a factor (of type II_1) if every non-trivial conjugacy class of G is infinite (for example, a non-abelian free group), and is the hyperfinite factor of type II_1 if in addition G is a union of finite subgroups (for example, the group of all permutations of the integers fixing all but a finite number of elements).
- The tensor product of two von Neumann algebras, or of a countable number with states, is a von Neumann algebra as described in the section above.
- The crossed product of a von Neumann algebra by a discrete (or more generally locally compact) group can be defined, and is a von Neumann algebra. Special cases are the **group-measure space construction** of Murray and von Neumann and **Krieger factors**.
- The von Neumann algebras of a measurable equivalence relation and a measurable groupoid can be defined. These examples generalise von Neumann group algebras and the group-measure space construction.

Applications

Von Neumann algebras have found applications in diverse areas of mathematics like knot theory, statistical mechanics, Quantum field theory, Local quantum physics, Free probability, Noncommutative geometry, representation theory, geometry, and probability.

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C*-algebra

C*-algebras (pronounced "C-star") are an important area of research in functional analysis, a branch of mathematics. The prototypical example of a C*-algebra is a complex algebra *A* of linear operators on a complex Hilbert space with two additional properties:

- *A* is a topologically closed set in the norm topology of operators.
- *A* is closed under the operation of taking adjoints of operators.

It is generally believed that C*-algebras were first considered primarily for their use in quantum mechanics to model algebras of physical observables. This line of research began with Werner Heisenberg's matrix mechanics and in a more mathematically developed form with Pascual Jordan around 1933. Subsequently John von Neumann attempted to establish a general framework for these algebras which culminated in a series of papers on rings of operators. These papers considered a special class of C*-algebras which are now known as von Neumann algebras.

Around 1943, the work of Israel Gelfand and Mark Naimark yielded an abstract characterisation of C*-algebras making no reference to operators.

C*-algebras are now an important tool in the theory of unitary representations of locally compact groups, and are also used in algebraic formulations of quantum mechanics.

Abstract characterization

We begin with the abstract characterization of C*-algebras given in the 1943 paper by Gelfand and Naimark.

A C*-algebra, *A*, is a Banach algebra over the field of complex numbers, together with a map, $* : A \to A$, called an involution. The image of an element *x* of *A* under the involution is written *x**. Involution has the following properties:

- For all *x*, *y* in *A*:

$$(x+y)^* = x^* + y^*$$
$$(xy)^* = y^* x^*$$

- For every λ in **C** and every *x* in *A*:

$$(\lambda x)^* = \overline{\lambda} x^*.$$

- For all *x* in *A*

$$(x^*)^* = x$$

- The **C*–identity** holds for all *x* in *A*:

$$\|x^* x\| = \|x\| \|x^*\|.$$

Note that the C* identity is equivalent to: for all *x* in *A*:

$$\|x x^*\| = \|x\| \|x^*\|.$$

This relation is equivalent to $\|x x^*\| = \|x\|^2$, which is sometimes called the B*-identity. For history behind the names C*- and B*-algebras, see the history section below.

The C*-identity is a very strong requirement. For instance, together with the spectral radius formula, it implies the C*-norm is uniquely determined by the algebraic structure:

$$\|x\|^2 = \|x^* x\| = \sup\{|\lambda| : x^* x - \lambda\, 1 \text{ is not invertible}\}.$$

A bounded linear map, $\pi : A \to B$, between C*-algebras *A* and *B* is called a ***-homomorphism** if

- For *x* and *y* in *A*

$$\pi(xy) = \pi(x)\pi(y)$$

- For x in A

$$\pi(x^*) = \pi(x)^*$$

In the case of C*-algebras, any *-homomorphism π between C*-algebras is non-expansive, i.e. bounded with norm ≤ 1. Furthermore, an injective *-homomorphism between C*-algebras is isometric. These are consequences of the C*-identity.

A bijective *-homomorphism π is called a **C*-isomorphism**, in which case A and B are said to be **isomorphic**.

Some history: B*-algebras and C*-algebras

The term B*-algebra was introduced by C. E. Rickart in 1946 to describe Banach *-algebras that satisfy the condition

- $\|x x^*\| = \|x\|^2$ for all x in the given B*-algebra. (B*-condition)

This condition automatically implies that the *-involution is isometric, that is, $\|x^*\| = \|x\|$. Hence $\|x x^*\| = \|x\| \|x^*\|$, and therefore, a B*-algebra is a C*-algebra. Conversely, the C*-condition implies the B*-condition. This is nontrivial, and can be proved without using the condition $\|x\|=\|x^*\|$. (For details, see R. S. Doran, V. A. Belfi, *Characterizations of C*-Algebras --- the Gelfand-Naimark Theorems*, CRC, 1986.)

For these reasons, the term B*-algebra is rarely used in current terminology, and has been replaced by the term 'C* algebra'.

The term C*-algebra was introduced by I. E. Segal in 1947 to describe norm-closed subalgebras of B(H), namely, the space of bounded operators on some Hilbert space H. 'C' stood for 'closed'.

Examples

Finite-dimensional C*-algebras

The algebra $M_n(\mathbf{C})$ of n-by-n matrices over **C** becomes a C*-algebra if we consider matrices as operators on the Euclidean space, $\mathbf{C}^n$, and use the operator norm $\|.\|$ on matrices. The involution is given by the conjugate transpose. More generally, one can consider finite direct sums of matrix algebras. In fact, all finite dimensional C*-algebras are of this form. The self-adjoint requirement means finite-dimensional C*-algebras are semisimple, from which fact one can deduce the following theorem of Artin–Wedderburn type:

Theorem. A finite-dimensional C*-algebra, A, is canonically isomorphic to a finite direct sum

$$A = \bigoplus_{e \in \min A} Ae$$

where min A is the set of minimal nonzero self-adjoint central projections of A.

Each C*-algebra, Ae, is isomorphic (in a noncanonical way) to the full matrix algebra $M_{\dim(e)}(\mathbf{C})$. The finite family indexed on min A given by $\{\dim(e)\}_e$ is called the *dimension vector* of A. This vector uniquely determines the isomorphism class of a finite-dimensional C*-algebra. In the language of K-theory, this vector is the positive cone of the K_0 group of A.

C*-algebras of operators

The prototypical example of a C*-algebra is the algebra *B(H)* of bounded (equivalently continuous) linear operators defined on a complex Hilbert space *H*; here x^* denotes the adjoint operator of the operator $x : H \to H$. In fact, every C*-algebra, *A*, is *-isomorphic to a norm-closed adjoint closed subalgebra of *B(H)* for a suitable Hilbert space, *H*; this is the content of the Gelfand–Naimark theorem.

Commutative C*-algebras

Let *X* be a locally compact Hausdorff space. The space $C_0(X)$ of complex-valued continuous functions on *X* that *vanish at infinity* (defined in the article on local compactness) form a commutative C*-algebra $C_0(X)$ under pointwise multiplication and addition. The involution is pointwise conjugation. $C_0(X)$ has a multiplicative unit element if and only if *X* is compact. As does any C*-algebra, $C_0(X)$ has an approximate identity. In the case of $C_0(X)$ this is immediate: consider the directed set of compact subsets of *X*, and for each compact *K* let f_K be a function of compact support which is identically 1 on *K*. Such functions exist by the Tietze extension theorem which applies to locally compact Hausdorff spaces. $\{f_K\}_K$ is an approximate identity.

The Gelfand representation states that every commutative C*-algebra is *-isomorphic to the algebra $C_0(X)$, where *X* is the space of characters equipped with the weak* topology. Furthermore if $C_0(X)$ is isomorphic to $C_0(Y)$ as C*-algebras, it follows that *X* and *Y* are homeomorphic. This characterization is one of the motivations for the noncommutative topology and noncommutative geometry programs.

C*-algebras of compact operators

Let *H* be a separable infinite-dimensional Hilbert space. The algebra *K*(*H*) of compact operators on *H* is a norm closed subalgebra of *B*(*H*). It is also closed under involution; hence it is a C*-algebra.

Concrete C*-algebras of compact operators admit a characterization similar to Wedderburn's theorem for finite dimensional C*-algebras.

Theorem. If *A* is a C*-subalgebra of *K*(*H*), then there exists Hilbert spaces $\{H_i\}_{i \in I}$ such that *A* is isomorphic to the following direct sum

$$\bigoplus_{i \in I} K(H_i),$$

where the (C*-)direct sum consists of elements (T_i) of the Cartesian product $\Pi\ K(H_i)$ with $\|T_i\| \to 0$.

Though *K*(*H*) does not have an identity element, a sequential approximate identity for *K*(*H*) can be easily displayed. To be specific, *H* is isomorphic to the space of square summable sequences l^2; we may assume that

$$H = \ell^2.$$

For each natural number *n* let H_n be the subspace of sequences of l^2 which vanish for indices

$$k \geq n$$

and let

$$e_n$$

be the orthogonal projection onto H_n. The sequence $\{e_n\}_n$ is an approximate identity for *K*(*H*).

K(*H*) is a two-sided closed ideal of *B*(*H*). For separable Hilbert spaces, it is the unique ideal. The quotient of *B*(*H*) by *K*(*H*) is the Calkin algebra.

C*-enveloping algebra

Given a B*-algebra A with an approximate identity, there is a unique (up to C*-isomorphism) C*-algebra **E**(A) and *-morphism π from A into **E**(A) which is universal, that is, every other B*-morphism $\pi' : A \to B$ factors uniquely through π. The algebra **E**(A) is called the **C*-enveloping algebra** of the B*-algebra A.

Of particular importance is the C*-algebra of a locally compact group G. This is defined as the enveloping C*-algebra of the group algebra of G. The C*-algebra of G provides context for general harmonic analysis of G in the case G is non-abelian. In particular, the dual of a locally compact group is defined to be the primitive ideal space of the group C*-algebra. See spectrum of a C*-algebra.

von Neumann algebras

von Neumann algebras, known as W* algebras before the 1960s, are a special kind of C*-algebra. They are required to be closed in the weak operator topology, which is weaker than the norm topology. Their study is a specialized area of functional analysis.

Properties of C*-algebras

C*-algebras have a large number of properties that are technically convenient. These properties can be established by use the continuous functional calculus or by reduction to commutative C*-algebras. In the latter case, we can use the fact that the structure of these is completely determined by the Gelfand isomorphism.

- The set of elements of a C*-algebra A of the form x^*x forms a closed convex cone. This cone is identical to the elements of the form $x x^*$. Elements of this cone are called *non-negative* (or sometimes *positive*, even though this terminology conflicts with its use for elements of **R**.)
- The set of self-adjoint elements of a C*-algebra A naturally has the structure of a partially ordered vector space; the ordering is usually denoted ≥. In this ordering, a self-adjoint element x of A satisfies $x \geq 0$ if and only if the spectrum of x is non-negative. Two self-adjoint elements x and y of A satisfy $x \geq y$ if $x - y \geq 0$.
- Any C*-algebra A has an approximate identity. In fact, there is a directed family $\{e_\lambda\}_{\lambda \in I}$ of self-adjoint elements of A such that

  $$xe_\lambda \to x$$
  $$0 \leq e_\lambda \leq e_\mu \leq 1 \quad \text{whenever } \lambda \leq \mu.$$

  In case A is separable, A has a sequential approximate identity. More generally, A will have a sequential approximate identity if and only if A contains a **strictly positive element**, i.e. a positive element h such that hAh is dense in A.
- Using approximate identities, one can show that the algebraic quotient of a C*-algebra by a closed proper two-sided ideal, with the natural norm, is a C*-algebra.
- Similarly, a closed two-sided ideal of a C*-algebra is itself a C*-algebra.

Type for C*-algebras

A C*-algebra **A** is of type I if and only if for all non-degenerate representations π of **A** the von Neumann algebra π(**A**)″ (that is, the bicommutant of π(**A**)) is a type I von Neumann algebra. In fact it is sufficient to consider only factor representations, i.e. representations π for which π(A)″ is a factor.

A locally compact group is said to be of type I if and only if its group C*-algebra is type I.

However, if a C*-algebra has non-type I representations, then by results of James Glimm it also has representations of type II and type III. Thus for C*-algebras and locally compact groups, it is only meaningful to speak of type I and non type I properties.

C*-algebras and quantum field theory

In quantum field theory, one typically describes a physical system with a C*-algebra A with unit element; the self-adjoint elements of A (elements x with $x^* = x$) are thought of as the *observables*, the measurable quantities, of the system. A *state* of the system is defined as a positive functional on A (a **C**-linear map $\varphi : A \to \mathbf{C}$ with $\varphi(u^* u) \geq 0$ for all $u \in A$) such that $\varphi(1) = 1$. The expected value of the observable x, if the system is in state φ, is then $\varphi(x)$.

See Local quantum physics.

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Quasi-Hopf algebra

A **quasi-Hopf algebra** is a generalization of a Hopf algebra, which was defined by the Russian mathematician Vladimir Drinfeld in 1989.

A *quasi-Hopf algebra* is a quasi-bialgebra $\mathcal{B}_{\mathcal{A}} = (\mathcal{A}, \Delta, \varepsilon, \Phi)$ for which there exist $\alpha, \beta \in \mathcal{A}$ and a bijective antihomomorphism *S* (antipode) of $\mathcal{A}$ such that

$$\sum_i S(b_i)\alpha c_i = \varepsilon(a)\alpha$$

$$\sum_i b_i\beta S(c_i) = \varepsilon(a)\beta$$

for all $a \in \mathcal{A}$ and where

$$\Delta(a) = \sum_i b_i \otimes c_i$$

and

$$\sum_i X_i\beta S(Y_i)\alpha Z_i = \mathbb{I},$$

$$\sum_j S(P_j)\alpha Q_j\beta S(R_j) = \mathbb{I}.$$

where the expansions for the quantities Φ and Φ^{-1} are given by

$$\Phi = \sum_i X_i \otimes Y_i \otimes Z_i$$

and

$$\Phi^{-1} = \sum_j P_j \otimes Q_j \otimes R_j.$$

As for a quasi-bialgebra, the property of being quasi-Hopf is preserved under twisting.

Usage

Quasi-Hopf algebras form the basis of the study of Drinfeld twists and the representations in terms of F-matrices associated with finite-dimensional irreducible representations of quantum affine algebra. F-matrices can be used to factorize the corresponding R-matrix. This leads to applications in Statistical mechanics, as quantum affine algebras, and their representations give rise to solutions of the Yang-Baxter equation, a solvability condition for various statistical models, allowing characteristics of the model to be deduced from its corresponding quantum affine algebra. The study of F-matrices has been applied to models such as the Heisenberg XXZ model in the framework of the algebraic Bethe ansatz. It provides a framework for solving two-dimensional integrable models by using the Quantum inverse scattering method.

References

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Quasitriangular Hopf algebra

In mathematics, a Hopf algebra, *H*, is **quasitriangular**[1] if there exists an invertible element, *R*, of $H \otimes H$ such that

- $R\ \Delta(x) = (T \circ \Delta)(x)\ R$ for all $x \in H$, where Δ is the coproduct on *H*, and the linear map $T : H \otimes H \to H \otimes H$ is given by $T(x \otimes y) = y \otimes x$,
- $(\Delta \otimes 1)(R) = R_{13}\ R_{23}$,
- $(1 \otimes \Delta)(R) = R_{13}\ R_{12}$,

where $R_{12} = \phi_{12}(R)$, $R_{13} = \phi_{13}(R)$, and $R_{23} = \phi_{23}(R)$, where $\phi_{12} : H \otimes H \to H \otimes H \otimes H$, $\phi_{13} : H \otimes H \to H \otimes H \otimes H$, and $\phi_{23} : H \otimes H \to H \otimes H \otimes H$, are algebra morphisms determined by

$$\phi_{12}(a \otimes b) = a \otimes b \otimes 1,$$
$$\phi_{13}(a \otimes b) = a \otimes 1 \otimes b,$$
$$\phi_{23}(a \otimes b) = 1 \otimes a \otimes b.$$

R is called the R-matrix.

As a consequence of the properties of quasitriangularity, the R-matrix, *R*, is a solution of the Yang-Baxter equation (and so a module *V* of *H* can be used to determine quasi-invariants of braids, knots and links). Also as a consequence of the properties of quasitriangularity, $(\epsilon \otimes 1)R = (1 \otimes \epsilon)R = 1 \in H$; moreover $R^{-1} = (S \otimes 1)(R)$, $R = (1 \otimes S)(R^{-1})$, and $(S \otimes S)(R) = R$. One may further show that the antipode *S* must be a linear isomorphism, and thus *S^2* is an automorphism. In fact, *S^2* is given by conjugating by an invertible element: $S(x) = uxu^{-1}$ where $u = m(S \otimes 1)R^{21}$ (cf. Ribbon Hopf algebras).

It is possible to construct a quasitriangular Hopf algebra from a Hopf algebra and its dual, using the Drinfel'd quantum double construction.

Twisting

The property of being a quasi-triangular Hopf algebra is preserved by twisting via an invertible element $F = \sum_i f^i \otimes f_i \in \mathcal{A} \otimes \mathcal{A}$ such that $(\varepsilon \otimes id)F = (id \otimes \varepsilon)F = 1$ and satisfying the cocycle condition

$$(F \otimes 1) \circ (\Delta \otimes id)F = (1 \otimes F) \circ (id \otimes \Delta)F$$

Furthermore, $u = \sum_i f^i S(f_i)$ is invertible and the twisted antipode is given by $S'(a) = uS(a)u^{-1}$, with the twisted comultiplication, R-matrix and co-unit change according to those defined for the quasi-triangular Quasi-Hopf algebra. Such a twist is known as an admissible (or Drinfel'd) twist.

Notes

[1] Montgomery & Schneider (2002), p. 72 (http://books.google.com/books?id=I3IK9U5Co_0C&pg=PA72&dq="Quasitriangular").

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Ribbon Hopf algebra

A **ribbon Hopf algebra** $(A, m, \Delta, u, \varepsilon, S, \mathcal{R}, \nu)$is a quasitriangular Hopf algebra which possess an invertible central element ν more commonly known as the ribbon element, such that the following conditions hold:

$$\nu^2 = uS(u),\ S(\nu) = \nu,\ \varepsilon(\nu) = 1$$

$$\Delta(\nu) = (\mathcal{R}_{21}\mathcal{R}_{12})^{-1}(\nu \otimes \nu)$$

where $u = m(S \otimes \mathrm{id})(\mathcal{R}_{21})$. Note that the element *u* exists for any quasitriangular Hopf algebra, and $uS(u)$ must always be central and satisfies $S(uS(u)) = uS(u), \varepsilon(uS(u)) = 1, \Delta(uS(u)) = (\mathcal{R}_{21}\mathcal{R}_{12})^{-2}(uS(u) \otimes uS(u))$, so that all that is required is that it have a central square root with the above properties.

Here

A is a vector space

m is the multiplication map $m : A \otimes A \to A$

Δ is the co-product map $\Delta : A \to A \otimes A$

u is the unit operator $u : \mathbb{C} \to A$

ε is the co-unit operator $\varepsilon : A \to \mathbb{C}$

S is the antipode $S : A \to A$

$\mathcal{R}$ is a universal R matrix

We assume that the underlying field K is $\mathbb{C}$

See also

- Quasitriangular Hopf algebra
- Quasi-triangular Quasi-Hopf algebra

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Quasi-triangular Quasi-Hopf algebra

A **quasi-triangular quasi-Hopf algebra** is a specialized form of a quasi-Hopf algebra defined by the Ukrainian mathematician Vladimir Drinfeld in 1989. It is also a generalized form of a quasi-triangular Hopf algebra.

A **quasi-triangular quasi-Hopf algebra** is a set $\mathcal{H}_{\mathcal{A}} = (\mathcal{A}, R, \Delta, \varepsilon, \Phi)$ where $\mathcal{B}_{\mathcal{A}} = (\mathcal{A}, \Delta, \varepsilon, \Phi)$ is a quasi-Hopf algebra and $R \in \mathcal{A} \otimes \mathcal{A}$ known as the R-matrix, is an invertible element such that

$$R\Delta(a) = \sigma \circ \Delta(a)R, a \in \mathcal{A}$$
$$\sigma : \mathcal{A} \otimes \mathcal{A} \to \mathcal{A} \otimes \mathcal{A}$$
$$x \otimes y \to y \otimes x$$

so that σ is the switch map and

$$(\Delta \otimes id)R = \Phi_{321} R_{13} \Phi_{132}^{-1} R_{23} \Phi_{123}$$
$$(id \otimes \Delta)R = \Phi_{231}^{-1} R_{13} \Phi_{213} R_{12} \Phi_{123}^{-1}$$

where $\Phi_{abc} = x_a \otimes x_b \otimes x_c$ and $\Phi_{123} = \Phi = x_1 \otimes x_2 \otimes x_3 \in \mathcal{A} \otimes \mathcal{A} \otimes \mathcal{A}$.

The quasi-Hopf algebra becomes *triangular* if in addition, $R_{21} R_{12} = 1$.

The twisting of $\mathcal{H}_{\mathcal{A}}$ by $F \in \mathcal{A} \otimes \mathcal{A}$ is the same as for a quasi-Hopf algebra, with the additional definition of the twisted *R*-matrix

A quasi-triangular (resp. triangular) quasi-Hopf algebra with $\Phi = 1$ is a quasi-triangular (resp. triangular) Hopf algebra as the latter two conditions in the definition reduce the conditions of quasi-triangularity of a Hopf algebra .

Similarly to the twisting properties of the quasi-Hopf algebra, the property of being quasi-triangular or triangular quasi-Hopf algebra is preserved by twisting.

See also

- Quasitriangular Hopf algebra
- Ribbon Hopf algebra

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Quantum inverse scattering method

Quantum inverse scattering method relates two different approaches: 1)Inverse scattering transform is a method of solving classical integrable differential equations of evolutionary type. Important concept is Lax representation. 2) Bethe ansatz is a method of solving quantum models in one space and one time dimension. Quantum inverse scattering method starts by quantization of Lax representation and reproduce results of Bethe ansatz. Actually it permits to rewrite Bethe ansatz in a new form: algebraic Bethe ansatz. This led to further progress in understanding of Heisenberg model (quantum), quantum Nonlinear Schrödinger equation (also known as Bose gas with delta interaction) and Hubbard model. Theory of correlation functions was developed. In mathematics in led to formulation of quantum groups. Especially interesting one is Yangian.

In mathematics, the **quantum inverse scattering method** is a method for solving integrable models in 1+1 dimensions introduced by L. D. Faddeev in about 1979.

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Grassmann algebra

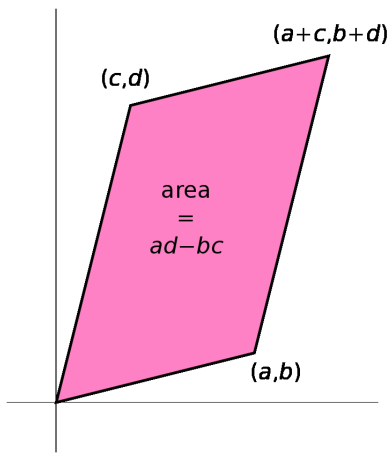
In mathematics, the **exterior product** or **wedge product** of vectors is an algebraic construction generalizing certain features of the cross product to higher dimensions. Like the cross product, and the scalar triple product, the exterior product of vectors is used in Euclidean geometry to study areas, volumes, and their higher-dimensional analogs. Also, like the cross product, the exterior product is alternating, meaning that $u \wedge u = 0$ for all vectors u, or equivalently[1] $u \wedge v = -v \wedge u$ for all vectors u and v. In linear algebra, the exterior product provides an abstract algebraic manner for describing the determinant and the minors of a linear transformation that is basis-independent, and is fundamentally related to ideas of rank and linear independence.

The **exterior algebra** (also known as the **Grassmann algebra**, after Hermann Grassmann[2]) of a given vector space V over a field K is the unital associative algebra $\Lambda(V)$ generated by the exterior product. It is widely used in contemporary geometry, especially differential geometry and algebraic geometry through the algebra of differential forms, as well as in multilinear algebra and related fields. In terms of category theory, the exterior algebra is a type of functor on vector spaces, given by a universal construction. The universal construction allows the exterior algebra to be defined, not just for vector spaces over a field, but also for modules over a commutative ring, and for other structures of interest. The exterior algebra is one example of a bialgebra, meaning that its dual space also possesses a product, and this dual product is compatible with the wedge product. This dual algebra is precisely the algebra of alternating multilinear forms on V, and the pairing between the exterior algebra and its dual is given by the interior product.

Motivating examples

Areas in the plane

The Cartesian plane $\mathbf{R}^2$ is a vector space equipped with a basis consisting of a pair of unit vectors



The area of a parallelogram in terms of the determinant of the matrix of coordinates of two of its vertices.

$$\mathbf{e}_1 = (1,0), \quad \mathbf{e}_2 = (0,1).$$

Suppose that

$$\mathbf{v} = v_1\mathbf{e}_1 + v_2\mathbf{e}_2, \quad \mathbf{w} = w_1\mathbf{e}_1 + w_2\mathbf{e}_2$$

are a pair of given vectors in $\mathbf{R}^2$, written in components. There is a unique parallelogram having **v** and **w** as two of its sides. The *area* of this parallelogram is given by the standard determinant formula:

$$A = \left|\det \begin{bmatrix}\mathbf{v} & \mathbf{w}\end{bmatrix}\right| = |v_1w_2 - v_2w_1|.$$

Consider now the exterior product of **v** and **w**:

$$\begin{aligned}\mathbf{v}\wedge\mathbf{w} &= (v_1\mathbf{e}_1 + v_2\mathbf{e}_2)\wedge(w_1\mathbf{e}_1 + w_2\mathbf{e}_2)\\ &= v_1w_1\mathbf{e}_1\wedge\mathbf{e}_1 + v_1w_2\mathbf{e}_1\wedge\mathbf{e}_2 + v_2w_1\mathbf{e}_2\wedge\mathbf{e}_1 + v_2w_2\mathbf{e}_2\wedge\mathbf{e}_2\\ &= (v_1w_2 - v_2w_1)\mathbf{e}_1\wedge\mathbf{e}_2\end{aligned}$$

where the first step uses the distributive law for the wedge product, and the last uses the fact that the wedge product is alternating, and in particular $\mathbf{e}_2 \wedge \mathbf{e}_1 = -\mathbf{e}_1 \wedge \mathbf{e}_2$. Note that the coefficient in this last expression is precisely the determinant of the matrix [**v** **w**]. The fact that this may be positive or negative has the intuitive meaning that **v** and **w** may be oriented in a counterclockwise or clockwise sense as the vertices of the parallelogram they define. Such an area is called the **signed area** of the parallelogram: the absolute value of the signed area is the ordinary area, and the sign determines its orientation.

The fact that this coefficient is the signed area is not an accident. In fact, it is relatively easy to see that the exterior product should be related to the signed area if one tries to axiomatize this area as an algebraic construct. In detail, if A(**v**,**w**) denotes the signed area of the parallelogram determined by the pair of vectors **v** and **w**, then A must satisfy the following properties:

1. A(a**v**,b**w**) = a b A(**v**,**w**) for any real numbers a and b, since rescaling either of the sides rescales the area by the same amount (and reversing the direction of one of the sides reverses the orientation of the parallelogram).
2. A(**v**,**v**) = 0, since the area of the degenerate parallelogram determined by **v** (i.e., a line segment) is zero.
3. A(**w**,**v**) = −A(**v**,**w**), since interchanging the roles of **v** and **w** reverses the orientation of the parallelogram.
4. A(**v** + a**w**,**w**) = A(**v**,**w**), since adding a multiple of **w** to **v** affects neither the base nor the height of the parallelogram and consequently preserves its area.
5. A($\mathbf{e}_1$, $\mathbf{e}_2$) = 1, since the area of the unit square is one.

With the exception of the last property, the wedge product satisfies the same formal properties as the area. In a certain sense, the wedge product generalizes the final property by allowing the area of a parallelogram to be compared to that of any "standard" chosen parallelogram. In other words, the exterior product in two-dimensions is a *basis-independent* formulation of area.[3]

Cross and triple products

For vectors in $\mathbf{R}^3$, the exterior algebra is closely related to the cross product and triple product. Using the standard basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$, the wedge product of a pair of vectors

$$\mathbf{u} = u_1\mathbf{e}_1 + u_2\mathbf{e}_2 + u_3\mathbf{e}_3$$

and

$$\mathbf{v} = v_1\mathbf{e}_1 + v_2\mathbf{e}_2 + v_3\mathbf{e}_3$$

is

$$\mathbf{u}\wedge\mathbf{v} = (u_1v_2 - u_2v_1)(\mathbf{e}_1\wedge\mathbf{e}_2) + (u_3v_1 - u_1v_3)(\mathbf{e}_3\wedge\mathbf{e}_1) + (u_2v_3 - u_3v_2)(\mathbf{e}_2\wedge\mathbf{e}_3)$$

where $\{\mathbf{e}_1 \wedge \mathbf{e}_2, \mathbf{e}_3 \wedge \mathbf{e}_1, \mathbf{e}_2 \wedge \mathbf{e}_3\}$ is the basis for the three-dimensional space $\Lambda^2(\mathbf{R}^3)$. This imitates the usual definition of the cross product of vectors in three dimensions.

Bringing in a third vector

$$\mathbf{w} = w_1\mathbf{e}_1 + w_2\mathbf{e}_2 + w_3\mathbf{e}_3,$$

the wedge product of three vectors is

$$\mathbf{u}\wedge\mathbf{v}\wedge\mathbf{w} = (u_1v_2w_3 + u_2v_3w_1 + u_3v_1w_2 - u_1v_3w_2 - u_2v_1w_3 - u_3v_2w_1)(\mathbf{e}_1\wedge\mathbf{e}_2\wedge\mathbf{e}_3)$$

where $\mathbf{e}_1 \wedge \mathbf{e}_2 \wedge \mathbf{e}_3$ is the basis vector for the one-dimensional space $\Lambda^3(\mathbf{R}^3)$. This imitates the usual definition of the triple product.

The cross product and triple product in three dimensions each admit both geometric and algebraic interpretations. The cross product **u**×**v** can be interpreted as a vector which is perpendicular to both **u** and **v** and whose magnitude is equal to the area of the parallelogram determined by the two vectors. It can also be interpreted as the vector consisting of the minors of the matrix with columns **u** and **v**. The triple product of **u**, **v**, and **w** is geometrically a (signed) volume. Algebraically, it is the determinant of the matrix with columns **u**, **v**, and **w**. The exterior product in three-dimensions allows for similar interpretations. In fact, in the presence of a positively oriented orthonormal basis, the exterior product generalizes these notions to higher dimensions.

Formal definitions and algebraic properties

The exterior algebra $\Lambda(V)$ over a vector space V is defined as the quotient algebra of the tensor algebra by the two-sided ideal I generated by all elements of the form $x \otimes x$ such that $x \in V$.[4] Symbolically,

$$\Lambda(V) := T(V)/I.$$

The wedge product $\wedge$ of two elements of $\Lambda(V)$ is defined by

$$\alpha \wedge \beta = \alpha \otimes \beta \pmod{I}.$$

Anticommutativity of the wedge product

The wedge product is *alternating* on elements of V, which means that $x \wedge x = 0$ for all $x \in V$. It follows that the product is also anticommutative on elements of V, for supposing that $x, y \in V$,

$$0 = (x+y) \wedge (x+y) = x \wedge x + x \wedge y + y \wedge x + y \wedge y = x \wedge y + y \wedge x$$

whence

$$x \wedge y = -y \wedge x.$$

More generally, if $x_1, x_2, ..., x_k$ are elements of V, and σ is a permutation of the integers $[1,...,k]$, then

$$x_{\sigma(1)} \wedge x_{\sigma(2)} \wedge \cdots \wedge x_{\sigma(k)} = \operatorname{sgn}(\sigma) x_1 \wedge x_2 \wedge \cdots \wedge x_k,$$

where $\operatorname{sgn}(\sigma)$ is the signature of the permutation σ.[5]

The exterior power

The kth **exterior power** of V, denoted $\Lambda^k(V)$, is the vector subspace of $\Lambda(V)$ spanned by elements of the form

$$x_1 \wedge x_2 \wedge \ldots \wedge x_k, \quad x_i \in V, i = 1, 2, \ldots, k.$$

If $\alpha \in \Lambda^k(V)$, then α is said to be a k-**multivector**. If, furthermore, α can be expressed as a wedge product of k elements of V, then α is said to be **decomposable**. Although decomposable multivectors span $\Lambda^k(V)$, not every element of $\Lambda^k(V)$ is decomposable. For example, in $\mathbf{R}^4$, the following 2-multivector is not decomposable:

$$\alpha = e_1 \wedge e_2 + e_3 \wedge e_4.$$

(This is in fact a symplectic form, since $\alpha \wedge \alpha \neq 0$.[6])

Basis and dimension

If the dimension of V is n and $\{e_1,\ldots,e_n\}$ is a basis of V, then the set

$$\{e_{i_1} \wedge e_{i_2} \wedge \cdots \wedge e_{i_k} \mid 1 \leq i_1 < i_2 < \cdots < i_k \leq n\}$$

is a basis for $\Lambda^k(V)$. The reason is the following: given any wedge product of the form

$$v_1 \wedge \cdots \wedge v_k$$

then every vector v_j can be written as a linear combination of the basis vectors e_i; using the bilinearity of the wedge product, this can be expanded to a linear combination of wedge products of those basis vectors. Any wedge product in which the same basis vector appears more than once is zero; any wedge product in which the basis vectors do not appear in the proper order can be reordered, changing the sign whenever two basis vectors change places. In general, the resulting coefficients of the basis k-vectors can be computed as the minors of the matrix that describes the vectors v_j in terms of the basis e_i.

By counting the basis elements, the dimension of $\Lambda^k(V)$ is the binomial coefficient $C(n,k)$. In particular, $\Lambda^k(V) = \{0\}$ for $k > n$.

Any element of the exterior algebra can be written as a sum of multivectors. Hence, as a vector space the exterior algebra is a direct sum

$$\Lambda(V) = \Lambda^0(V) \oplus \Lambda^1(V) \oplus \Lambda^2(V) \oplus \cdots \oplus \Lambda^n(V)$$

(where by convention $\Lambda^0(V) = K$ and $\Lambda^1(V) = V$), and therefore its dimension is equal to the sum of the binomial coefficients, which is 2^n.

Rank of a multivector

If $\alpha \in \Lambda^k(V)$, then it is possible to express α as a linear combination of decomposable multivectors:

$$\alpha = \alpha^{(1)} + \alpha^{(2)} + \cdots + \alpha^{(s)}$$

where each $\alpha^{(i)}$ is decomposable, say

$$\alpha^{(i)} = \alpha_1^{(i)} \wedge \cdots \wedge \alpha_k^{(i)}, \quad i = 1, 2, \ldots, s.$$

The **rank** of the multivector α is the minimal number of decomposable multivectors in such an expansion of α. This is similar to the notion of tensor rank.

Rank is particularly important in the study of 2-multivectors (Sternberg 1974, §III.6) (Bryant et al. 1991). The rank of a 2-multivector α can be identified with half the rank of the matrix of coefficients of α in a basis. Thus if e_i is a basis for V, then α can be expressed uniquely as

$$\alpha = \sum_{i,j} a_{ij} e_i \wedge e_j$$

where $a_{ij} = -a_{ji}$ (the matrix of coefficients is skew-symmetric). The rank of the matrix a_{ij} is therefore even, and is twice the rank of the form α.

In characteristic 0, the 2-multivector α has rank p if and only if

$$\underbrace{\alpha \wedge \cdots \wedge \alpha}_{p} \neq 0$$

and

$$\underbrace{\alpha \wedge \cdots \wedge \alpha}_{p+1} = 0.$$

Graded structure

The wedge product of a k-multivector with a p-multivector is a $(k+p)$-multivector, once again invoking bilinearity. As a consequence, the direct sum decomposition of the preceding section

$$\Lambda(V) = \Lambda^0(V) \oplus \Lambda^1(V) \oplus \Lambda^2(V) \oplus \cdots \oplus \Lambda^n(V)$$

gives the exterior algebra the additional structure of a graded algebra. Symbolically,

$$\left(\Lambda^k(V)\right) \wedge \left(\Lambda^p(V)\right) \subset \Lambda^{k+p}(V).$$

Moreover, the wedge product is graded anticommutative, meaning that if $\alpha \in \Lambda^k(V)$ and $\beta \in \Lambda^p(V)$, then

$$\alpha \wedge \beta = (-1)^{kp} \beta \wedge \alpha.$$

In addition to studying the graded structure on the exterior algebra, Bourbaki (1989) studies additional graded structures on exterior algebras, such as those on the exterior algebra of a graded module (a module that already carries its own gradation).

Universal property

Let V be a vector space over the field K. Informally, multiplication in $\Lambda(V)$ is performed by manipulating symbols and imposing a distributive law, an associative law, and using the identity $v \wedge v = 0$ for $v \in V$. Formally, $\Lambda(V)$ is the "most general" algebra in which these rules hold for the multiplication, in the sense that any unital associative K-algebra containing V with alternating multiplication on V must contain a homomorphic image of $\Lambda(V)$. In other words, the exterior algebra has the following universal property:[7]

Given any unital associative K-algebra A and any K-linear map $j : V \to A$ such that $j(v)j(v) = 0$ for every v in V, then there exists *precisely one* unital algebra homomorphism $f : \Lambda(V) \to A$ such that $j(v) = f(i(v))$ for all v in V.

$$\begin{tikzcd} V \arrow[r, "i"] \arrow[dr, "j"'] & \Lambda(V) \arrow[d, dashed, "f"] \\ & A \end{tikzcd}$$

To construct the most general algebra that contains V and whose multiplication is alternating on V, it is natural to start with the most general algebra that contains V, the tensor algebra $T(V)$, and then enforce the alternating property by taking a suitable quotient. We thus take the two-sided ideal I in $T(V)$ generated by all elements of the form $v \otimes v$ for v in V, and define $\Lambda(V)$ as the quotient

$$\Lambda(V) = T(V)/I$$

(and use $\wedge$ as the symbol for multiplication in $\Lambda(V)$). It is then straightforward to show that $\Lambda(V)$ contains V and satisfies the above universal property.

As a consequence of this construction, the operation of assigning to a vector space V its exterior algebra $\Lambda(V)$ is a functor from the category of vector spaces to the category of algebras.

Rather than defining $\Lambda(V)$ first and then identifying the exterior powers $\Lambda^k(V)$ as certain subspaces, one may alternatively define the spaces $\Lambda^k(V)$ first and then combine them to form the algebra $\Lambda(V)$. This approach is often used in differential geometry and is described in the next section.

Generalizations

Given a commutative ring R and an R-module M, we can define the exterior algebra $\Lambda(M)$ just as above, as a suitable quotient of the tensor algebra $\mathbf{T}(M)$. It will satisfy the analogous universal property. Many of the properties of $\Lambda(M)$ also require that M be a projective module. Where finite-dimensionality is used, the properties further require that M be finitely generated and projective. Generalizations to the most common situations can be found in (Bourbaki 1989).

Exterior algebras of vector bundles are frequently considered in geometry and topology. There are no essential differences between the algebraic properties of the exterior algebra of finite-dimensional vector bundles and those of the exterior algebra of finitely-generated projective modules, by the Serre-Swan theorem. More general exterior algebras can be defined for sheaves of modules.

Duality

Alternating operators

Given two vector spaces V and X, an **alternating operator** (or *anti-symmetric operator*) from V^k to X is a multilinear map

$$f : V^k \to X$$

such that whenever $v_1,\ldots,v_k$ are linearly dependent vectors in V, then

$$f(v_1, \ldots, v_k) = 0$$

A well-known example is the determinant, an alternating operator from $(K^n)^n$ to K.

The map

$$w : V^k \to \wedge^k(V)$$

which associates to k vectors from V their wedge product, i.e. their corresponding k-vector, is also alternating. In fact, this map is the "most general" alternating operator defined on V^k: given any other alternating operator $f : V^k \to X$, there exists a unique linear map $\varphi\colon \Lambda^k(V) \to X$ with $f = \varphi \circ w$. This universal property characterizes the space $\Lambda^k(V)$ and can serve as its definition.

Alternating multilinear forms

The above discussion specializes to the case when $X = K$, the base field. In this case an alternating multilinear function

$$f : V^k \to K$$

is called an **alternating multilinear form**. The set of all alternating multilinear forms is a vector space, as the sum of two such maps, or the product of such a map with a scalar, is again alternating. By the universal property of the exterior power, the space of alternating forms of degree k on V is naturally isomorphic with the dual vector space $(\Lambda^k V)^*$. If V is finite-dimensional, then the latter is naturally isomorphic to $\Lambda^k(V^*)$. In particular, the dimension of the space of anti-symmetric maps from V^k to K is the binomial coefficient n choose k.

Under this identification, the wedge product takes a concrete form: it produces a new anti-symmetric map from two given ones. Suppose $\omega : V^k \to K$ and $\eta : V^m \to K$ are two anti-symmetric maps. As in the case of tensor products of multilinear maps, the number of variables of their wedge product is the sum of the numbers of their variables. It is defined as follows:

$$\omega \wedge \eta = \frac{(k+m)!}{k!\,m!}\mathrm{Alt}(\omega \otimes \eta)$$

where the alternation Alt of a multilinear map is defined to be the signed average of the values over all the permutations of its variables:

$$\mathrm{Alt}(\omega)(x_1,\ldots,x_k) = \frac{1}{k!}\sum_{\sigma\in S_k} \mathrm{sgn}(\sigma)\,\omega(x_{\sigma(1)},\ldots,x_{\sigma(k)}).$$

This definition of the wedge product is well-defined even if the field *K* has finite characteristic, if one considers an equivalent version of the above that does not use factorials or any constants:

$$\omega\wedge\eta(x_1,\ldots,x_{k+m}) = \sum_{\sigma\in Sh_{k,m}} \mathrm{sgn}(\sigma)\,\omega(x_{\sigma(1)},\ldots,x_{\sigma(k)})\eta(x_{\sigma(k+1)},\ldots,x_{\sigma(k+m)}),$$

where here $Sh_{k,m} \subset S_{k+m}$ is the subset of (k,m) shuffles: permutations σ of the set $\{1,2,\ldots,k+m\}$ such that $\sigma(1) < \sigma(2) < \ldots < \sigma(k)$, and $\sigma(k+1) < \sigma(k+2) < \ldots < \sigma(k+m)$.[8]

Bialgebra structure

In formal terms, there is a correspondence between the graded dual of the graded algebra Λ(*V*) and alternating multilinear forms on *V*. The wedge product of multilinear forms defined above is dual to a coproduct defined on Λ(*V*), giving the structure of a coalgebra.

The **coproduct** is a linear function $\Delta : \Lambda(V) \to \Lambda(V) \otimes \Lambda(V)$ given on decomposable elements by

$$\Delta(x_1\wedge\cdots\wedge x_k) = \sum_{p=0}^{k} \sum_{\sigma\in Sh_{p,k-p}} \mathrm{sgn}(\sigma)(x_{\sigma(1)}\wedge\cdots\wedge x_{\sigma(p)})\otimes(x_{\sigma(p+1)}\wedge\cdots\wedge x_{\sigma(k)}).$$

For example,

$$\Delta(x_1) = 1\otimes x_1 + x_1\otimes 1,$$
$$\Delta(x_1\wedge x_2) = 1\otimes(x_1\wedge x_2) + x_1\otimes x_2 - x_2\otimes x_1 + (x_1\wedge x_2)\otimes 1.$$

This extends by linearity to an operation defined on the whole exterior algebra. In terms of the coproduct, the wedge product on the dual space is just the graded dual of the coproduct:

$$(\alpha\wedge\beta)(x_1\wedge\ldots\wedge x_k) = (\alpha\otimes\beta)\left(\Delta(x_1\wedge\ldots\wedge x_k)\right)$$

where the tensor product on the right-hand side is of multilinear linear maps (extended by zero on elements of incompatible homogeneous degree: more precisely, $\alpha\wedge\beta = \varepsilon \circ (\alpha\otimes\beta) \circ \Delta$, where ε is the counit, as defined presently).

The **counit** is the homomorphism $\varepsilon : \Lambda(V) \to K$ which returns the 0-graded component of its argument. The coproduct and counit, along with the wedge product, define the structure of a bialgebra on the exterior algebra.

With an **antipode** defined on homogeneous elements by $S(x) = (-1)^{\deg x}x$, the exterior algebra is furthermore a Hopf algebra.[9]

Interior product

Suppose that *V* is finite-dimensional. If *V** denotes the dual space to the vector space *V*, then for each $\alpha \in V^*$, it is possible to define an antiderivation on the algebra Λ(V),

$$i_\alpha : \Lambda^k V \to \Lambda^{k-1}V.$$

This derivation is called the **interior product** with α, or sometimes the **insertion operator**, or **contraction** by α.

Suppose that $\mathbf{w} \in \Lambda^k V$. Then **w** is a multilinear mapping of V^* to *K*, so it is defined by its values on the *k*-fold Cartesian product $V^* \times V^* \times \ldots \times V^*$. If $u_1, u_2, \ldots, u_{k-1}$ are *k-1* elements of V^*, then define

$$(i_\alpha\mathbf{w})(u_1,u_2\ldots,u_{k-1}) = \mathbf{w}(\alpha,u_1,u_2,\ldots,u_{k-1}).$$

Additionally, let $i_\alpha f = 0$ whenever *f* is a pure scalar (i.e., belonging to $\Lambda^0 V$).

Axiomatic characterization and properties

The interior product satisfies the following properties:

1. For each k and each $\alpha \in V^*$,

$$i_\alpha : \Lambda^k V \to \Lambda^{k-1} V.$$

(By convention, $\Lambda^{-1} = 0$.)

2. If v is an element of V ($= \Lambda^1 V$), then $i_\alpha v = \alpha(v)$ is the dual pairing between elements of V and elements of V^*.
3. For each $\alpha \in V^*$, i_α is a graded derivation of degree −1:

$$i_\alpha(a \wedge b) = (i_\alpha a) \wedge b + (-1)^{\deg a} a \wedge (i_\alpha b).$$

In fact, these three properties are sufficient to characterize the interior product as well as define it in the general infinite-dimensional case.

Further properties of the interior product include:

- $i_\alpha \circ i_\alpha = 0.$
- $i_\alpha \circ i_\beta = -i_\beta \circ i_\alpha.$

Hodge duality

Suppose that V has finite dimension n. Then the interior product induces a canonical isomorphism of vector spaces

$$\Lambda^k(V^*) \otimes \Lambda^n(V) \to \Lambda^{n-k}(V).$$

In the geometrical setting, a non-zero element of the top exterior power $\Lambda^n(V)$ (which is a one-dimensional vector space) is sometimes called a **volume form** (or **orientation form**, although this term may sometimes lead to ambiguity.) Relative to a given volume form σ, the isomorphism is given explicitly by

$$\alpha \in \Lambda^k(V^*) \mapsto i_\alpha \sigma \in \Lambda^{n-k}(V).$$

If, in addition to a volume form, the vector space V is equipped with an inner product identifying V with V^*, then the resulting isomorphism is called the **Hodge dual** (or more commonly the **Hodge star operator**)

$$* : \Lambda^k(V) \to \Lambda^{n-k}(V).$$

The composite of $*$ with itself maps $\Lambda^k(V) \to \Lambda^k(V)$ and is always a scalar multiple of the identity map. In most applications, the volume form is compatible with the inner product in the sense that it is a wedge product of an orthonormal basis of V. In this case,

$$* \circ * : \Lambda^k(V) \to \Lambda^k(V) = (-1)^{k(n-k)+q} I$$

where I is the identity, and the inner product has metric signature (p,q) — p plusses and q minuses.

Inner product

For V a finite-dimensional space, an inner product on V defines an isomorphism of V with V^*, and so also an isomorphism of $\Lambda^k V$ with $(\Lambda^k V)^*$. The pairing between these two spaces also takes the form of an inner product. On decomposable k-multivectors,

$$\langle v_1 \wedge \cdots \wedge v_k, w_1 \wedge \cdots \wedge w_k \rangle = \det(\langle v_i, w_j \rangle),$$

the determinant of the matrix of inner products. In the special case $v_i = w_i$, the inner product is the square norm of the multivector, given by the determinant of the Gramian matrix $(\langle v_i, v_j \rangle)$. This is then extended bilinearly (or sesquilinearly in the complex case) to a non-degenerate inner product on $\Lambda^k V$. If e_i, $i=1,2,\ldots,n$, form an orthonormal basis of V, then the vectors of the form

$$e_{i_1} \wedge \cdots \wedge e_{i_k}, \quad i_1 < \cdots < i_k,$$

constitute an orthonormal basis for $\Lambda^k(V)$.

With respect to the inner product, exterior multiplication and the interior product are mutually adjoint. Specifically, for $\mathbf{v} \in \Lambda^{k-1}(V)$, $\mathbf{w} \in \Lambda^{k}(V)$, and $x \in V$,

$$\langle x \wedge \mathbf{v}, \mathbf{w} \rangle = \langle \mathbf{v}, i_{x^\flat} \mathbf{w} \rangle$$

where $x^\flat \in V^*$ is the linear functional defined by

$$x^\flat(y) = \langle x, y \rangle$$

for all $y \in V$. This property completely characterizes the inner product on the exterior algebra.

Functoriality

Suppose that V and W are a pair of vector spaces and $f : V \to W$ is a linear transformation. Then, by the universal construction, there exists a unique homomorphism of graded algebras

$$\Lambda(f) : \Lambda(V) \to \Lambda(W)$$

such that

$$\Lambda(f)|_{\Lambda^1(V)} = f : V = \Lambda^1(V) \to W = \Lambda^1(W).$$

In particular, $\Lambda(f)$ preserves homogeneous degree. The k-graded components of $\Lambda(f)$ are given on decomposable elements by

$$\Lambda(f)(x_1 \wedge \ldots \wedge x_k) = f(x_1) \wedge \ldots \wedge f(x_k).$$

Let

$$\Lambda^k(f) = \Lambda(f)_{\Lambda^k(V)} : \Lambda^k(V) \to \Lambda^k(W).$$

The components of the transformation $\Lambda(k)$ relative to a basis of V and W is the matrix of $k \times k$ minors of f. In particular, if $V = W$ and V is of finite dimension n, then $\Lambda^n(f)$ is a mapping of a one-dimensional vector space Λ^n to itself, and is therefore given by a scalar: the determinant of f.

Exactness

If

$$0 \to U \to V \to W \to 0$$

is a short exact sequence of vector spaces, then

$$0 \to \Lambda^1(U) \wedge \Lambda(V) \to \Lambda(V) \to \Lambda(W) \to 0$$

is an exact sequence of graded vector spaces[10] as is

$$0 \to \Lambda(U) \to \Lambda(V).$$ [11]

Direct sums

In particular, the exterior algebra of a direct sum is isomorphic to the tensor product of the exterior algebras:

$$\Lambda(V \oplus W) = \Lambda(V) \otimes \Lambda(W).$$

This is a graded isomorphism; i.e.,

$$\Lambda^k(V \oplus W) = \bigoplus_{p+q=k} \Lambda^p(V) \otimes \Lambda^q(W).$$

Slightly more generally, if

$$0 \to U \to V \to W \to 0$$

is a short exact sequence of vector spaces then $\Lambda^k(V)$ has a filtration

$$0 = F^0 \subseteq F^1 \subseteq \cdots \subseteq F^k \subseteq F^{k+1} = \Lambda^k(V)$$

with quotients : $F^{p+1}/F^p = \Lambda^{k-p}(U) \otimes \Lambda^p(W)$. In particular, if U is 1-dimensional then

$$0 \to U \otimes \Lambda^{k-1}(W) \to \Lambda^k(V) \to \Lambda^k(W) \to 0$$

is exact, and if *W* is 1-dimensional then

$$0 \to \Lambda^k(U) \to \Lambda^k(V) \to \Lambda^{k-1}(U) \otimes W \to 0$$

is exact.[12]

The alternating tensor algebra

If *K* is a field of characteristic 0,[13] then the exterior algebra of a vector space *V* can be canonically identified with the vector subspace of T(*V*) consisting of antisymmetric tensors. Recall that the exterior algebra is the quotient of T(*V*) by the ideal *I* generated by $x \otimes x$.

Let $T^r(V)$ be the space of homogeneous tensors of degree *r*. This is spanned by decomposable tensors

$$v_1 \otimes \ldots \otimes v_r, \quad v_i \in V.$$

The **antisymmetrization** (or sometimes the **skew-symmetrization**) of a decomposable tensor is defined by

$$\mathrm{Alt}(v_1 \otimes \cdots \otimes v_r) = \frac{1}{r!} \sum_{\sigma \in \mathfrak{S}_r} \mathrm{sgn}(\sigma) v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(r)}$$

where the sum is taken over the symmetric group of permutations on the symbols {1,...,*r*}. This extends by linearity and homogeneity to an operation, also denoted by Alt, on the full tensor algebra T(*V*). The image Alt(T(*V*)) is the **alternating tensor algebra**, denoted A(*V*). This is a vector subspace of T(*V*), and it inherits the structure of a graded vector space from that on T(*V*). It carries an associative graded product $\widehat{\otimes}$ defined by

$$t \widehat{\otimes} s = \mathrm{Alt}(t \otimes s).$$

Although this product differs from the tensor product, the kernel of *Alt* is precisely the ideal *I* (again, assuming that *K* has characteristic 0), and there is a canonical isomorphism

$$A(V) \cong \Lambda(V).$$

Index notation

Suppose that *V* has finite dimension *n*, and that a basis $\mathbf{e}_1, \ldots, \mathbf{e}_n$ of *V* is given. then any alternating tensor $t \in A^r(V) \subset T^r(V)$ can be written in index notation as

$$t = t^{i_1 i_2 \ldots i_r} \, \mathbf{e}_{i_1} \otimes \mathbf{e}_{i_2} \otimes \cdots \otimes \mathbf{e}_{i_r}$$

where $t^{i_1 \cdots i_r}$ is completely antisymmetric in its indices.

The wedge product of two alternating tensors *t* and *s* of ranks *r* and *p* is given by

$$t \widehat{\otimes} s = \frac{1}{(r+p)!} \sum_{\sigma \in \mathfrak{S}_{r+p}} \mathrm{sgn}(\sigma) t^{i_{\sigma(1)} \cdots i_{\sigma(r)}} s^{i_{\sigma(r+1)} \cdots i_{\sigma(r+p)}} \mathbf{e}_{i_1} \otimes \mathbf{e}_{i_2} \otimes \cdots \otimes \mathbf{e}_{i_{r+p}}.$$

The components of this tensor are precisely the skew part of the components of the tensor product $s \otimes t$, denoted by square brackets on the indices:

$$(t \widehat{\otimes} s)^{i_1 \ldots i_{r+p}} = t^{[i_1 \ldots i_r} s^{i_{r+1} \ldots i_{r+p}]}.$$

The interior product may also be described in index notation as follows. Let $t = t^{i_0 i_1 \ldots i_{r-1}}$ be an antisymmetric tensor of rank *r*. Then, for $\alpha \in V^*$, $i_\alpha \mathbf{t}$ is an alternating tensor of rank *r*-1, given by

$$(i_\alpha t)^{i_1 \ldots i_{r-1}} = r \sum_{j=0}^{n} \alpha_j t^{j i_1 \ldots i_{r-1}}.$$

where *n* is the dimension of *V*.

Applications

Linear geometry

The decomposable *k*-vectors have geometric interpretations: the bivector $u \wedge v$ represents the plane spanned by the vectors, "weighted" with a number, given by the area of the oriented parallelogram with sides *u* and *v*. Analogously, the 3-vector $u \wedge v \wedge w$ represents the spanned 3-space weighted by the volume of the oriented parallelepiped with edges *u*, *v*, and *w*.

Projective geometry

Decomposable *k*-vectors in $\Lambda^k V$ correspond to weighted *k*-dimensional subspaces of *V*. In particular, the Grassmannian of *k*-dimensional subspaces of *V*, denoted $Gr_k(V)$, can be naturally identified with an algebraic subvariety of the projective space $\mathbf{P}(\Lambda^k V)$. This is called the Plücker embedding.

Differential geometry

The exterior algebra has notable applications in differential geometry, where it is used to define differential forms. A differential form at a point of a differentiable manifold is an alternating multilinear form on the tangent space at the point. Equivalently, a differential form of degree *k* is a linear functional on the *k*-th exterior power of the tangent space. As a consequence, the wedge product of multilinear forms defines a natural wedge product for differential forms. Differential forms play a major role in diverse areas of differential geometry.

In particular, the exterior derivative gives the exterior algebra of differential forms on a manifold the structure of a differential algebra. The exterior derivative commutes with pullback along smooth mappings between manifolds, and it is therefore a natural differential operator. The exterior algebra of differential forms, equipped with the exterior derivative, is a differential complex whose cohomology is called the de Rham cohomology of the underlying manifold and plays a vital role in the algebraic topology of differentiable manifolds.

Representation theory

In representation theory, the exterior algebra is one of the two fundamental Schur functors on the category of vector spaces, the other being the symmetric algebra. Together, these constructions are used to generate the irreducible representations of the general linear group; see fundamental representation.

Physics

The exterior algebra is an archetypal example of a superalgebra, which plays a fundamental role in physical theories pertaining to fermions and supersymmetry. For a physical discussion, see Grassmann number. For various other applications of related ideas to physics, see superspace and supergroup (physics).

Lie algebra homology

Let *L* be a Lie algebra over a field *k*, then it is possible to define the structure of a chain complex on the exterior algebra of *L*. This is a *k*-linear mapping

$$\partial : \Lambda^{p+1}L \to \Lambda^p L$$

defined on decomposable elements by

$$\partial(x_1\wedge\cdots\wedge x_{p+1}) = \frac{1}{p+1}\sum_{j<\ell}(-1)^{j+\ell+1}[x_j, x_\ell]\wedge x_1\wedge\cdots\wedge\hat{x}_j\wedge\cdots\wedge\hat{x}_\ell\wedge\cdots\wedge x_{p+1}.$$

The Jacobi identity holds if and only if $\partial\partial = 0$, and so this is a necessary and sufficient condition for an anticommutative nonassociative algebra *L* to be a Lie algebra. Moreover, in that case ΛL is a chain complex with boundary operator ∂. The homology associated to this complex is the Lie algebra homology.

Homological algebra

The exterior algebra is the main ingredient in the construction of the Koszul complex, a fundamental object in homological algebra.

History

The exterior algebra was first introduced by Hermann Grassmann in 1844 under the blanket term of *Ausdehnungslehre*, or *Theory of Extension*.[14] This referred more generally to an algebraic (or axiomatic) theory of extended quantities and was one of the early precursors to the modern notion of a vector space. Saint-Venant also published similar ideas of exterior calculus for which he claimed priority over Grassmann.[15]

The algebra itself was built from a set of rules, or axioms, capturing the formal aspects of Cayley and Sylvester's theory of multivectors. It was thus a *calculus*, much like the propositional calculus, except focused exclusively on the task of formal reasoning in geometrical terms.[16] In particular, this new development allowed for an *axiomatic* characterization of dimension, a property that had previously only been examined from the coordinate point of view.

The import of this new theory of vectors and multivectors was lost to mid 19th century mathematicians,[17] until being thoroughly vetted by Giuseppe Peano in 1888. Peano's work also remained somewhat obscure until the turn of the century, when the subject was unified by members of the French geometry school (notably Henri Poincaré, Élie Cartan, and Gaston Darboux) who applied Grassmann's ideas to the calculus of differential forms.

A short while later, Alfred North Whitehead, borrowing from the ideas of Peano and Grassmann, introduced his universal algebra. This then paved the way for the 20th century developments of abstract algebra by placing the axiomatic notion of an algebraic system on a firm logical footing.

Notes

[1] Provided the characteristic is different from 2.

[2] Grassmann (1844) introduced these as *extended* algebras (cf. Clifford 1878). He used the word *äußere* (literally translated as *outer*, or *exterior*) only to indicate the *produkt* he defined, which is nowadays conventionally called *exterior product*, probably to distinguish it from the *outer product* as defined in modern linear algebra.

[3] This axiomatization of areas is due to Leopold Kronecker and Karl Weierstrass; see Bourbaki (1989, Historical Note). For a modern treatment, see MacLane & Birkhoff (1999, Theorem IX.2.2). For an elementary treatment, see Strang (1993, Chapter 5).

[4] This definition is a standard one. See, for instance, MacLane & Birkhoff (1999).

[5] A proof of this can be found in more generality in Bourbaki (1989).

[6] See Sternberg (1964, §III.6).

[7] See Bourbaki (1989, III.7.1), and MacLane & Birkhoff (1999, Theorem XVI.6.8). More detail on universal properties in general can be found in MacLane & Birkhoff (1999, Chapter VI), and throughout the works of Bourbaki.

[8] Some conventions, particularly in physics, define the wedge product as

$$\omega \wedge \eta = \mathrm{Alt}(\omega \otimes \eta).$$

This convention is not adopted here, but is discussed in connection with alternating tensors.

[9] Indeed, the exterior algebra of V is the enveloping algebra of the abelian Lie superalgebra structure on V.

[10] This part of the statement also holds in greater generality if V and W are modules over a commutative ring: That Λ converts epimorphisms to epimorphisms. See Bourbaki (1989, Proposition 3, III.7.2).

[11] This statement generalizes only to the case where V and W are projective modules over a commutative ring. Otherwise, it is generally not the case that Λ converts monomorphisms to monomorphisms. See Bourbaki (1989, Corollary to Proposition 12, III.7.9).

[12] Such a filtration also holds for vector bundles, and projective modules over a commutative ring. This is thus more general than the result quoted above for direct sums, since not every short exact sequence splits in other abelian categories.

[13] See Bourbaki (1989, III.7.5) for generalizations.

[14] Kannenberg (2000) published a translation of Grassmann's work in English; he translated *Ausdehnungslehre* as *Extension Theory*.

[15] J Itard, Biography in Dictionary of Scientific Biography (New York 1970-1990).

[16] Authors have in the past referred to this calculus variously as the *calculus of extension* (Whitehead 1898; Forder 1941), or *extensive algebra* (Clifford 1878), and recently as *extended vector algebra* (Browne 2007).

[17] Bourbaki 1989, p. 661.

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- Bourbaki, Nicolas (1989), *Elements of mathematics, Algebra I*, Springer-Verlag, ISBN 3-540-64243-9

  This is the *main mathematical reference* for the article. It introduces the exterior algebra of a module over a commutative ring (although this article specializes primarily to the case when the ring is a field), including a discussion of the universal property, functoriality, duality, and the bialgebra structure. See chapters III.7 and III.11.

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  Chapter XVI sections 6-10 give a more elementary account of the exterior algebra, including duality, determinants and minors, and alternating forms.

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Other references and further reading

- Browne, J.M. (2007), *Grassmann algebra - Exploring applications of Extended Vector Algebra with Mathematica*, Published on line (http://www.grassmannalgebra.info/grassmannalgebra/book/index.htm)

  An introduction to the exterior algebra, and geometric algebra, with a focus on applications. Also includes a history section and bibliography.

- Spivak, Michael (1965), *Calculus on manifolds*, Addison-Wesley, ISBN 978-0805390216

  Includes applications of the exterior algebra to differential forms, specifically focused on integration and Stokes's theorem. The notation $\Lambda^k V$ in this text is used to mean the space of alternating *k*-forms on *V*; i.e., for Spivak $\Lambda^k V$ is what this article would call $\Lambda^k V^*$. Spivak discusses this in Addendum 4.

- Strang, G. (1993), *Introduction to linear algebra*, Wellesley-Cambridge Press, ISBN 978-0961408855

  Includes an elementary treatment of the axiomatization of determinants as signed areas, volumes, and higher-dimensional volumes.

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  Chapter 6: Exterior algebra and differential calculus, pages 205-38. This textbook in multivariate calculus introduces the exterior algebra of differential forms adroitly into the calculus sequence for colleges.

- Winitzki, S. (2010), *Linear Algebra via Exterior Products*, Published on line (http://sites.google.com/site/winitzki/linalg)

  An introduction to the coordinate-free approach in basic finite-dimensional linear algebra, using exterior products.

Supergroup

Supergroup or **super group** may refer to:

- Supergroup (music), a music group formed by artists who are already notable or respected in their fields
- Supergroup (physics), a generalization of groups, used in the study of supersymmetry
- Supergroup (City of Heroes), the term for player guilds in the *City of Heroes* MMORPG
- SuperGroup plc, a British company
- *Supergroup* (TV series), a VH1 reality show
- Supergroup, a geological unit
- Super-group, a team of superheroes who work together
- Supergroup, a rarely used term in mathematics for the counterpart of a subgroup
- In L-carrier, a multiplexed group of Channel Groups

Superalgebra

In mathematics and theoretical physics, a **superalgebra** is a $\mathbf{Z}_2$-graded algebra. [1] That is, it is an algebra over a commutative ring or field with a decomposition into "even" and "odd" pieces and a multiplication operator that respects the grading.

The prefix *super-* comes from the theory of supersymmetry in theoretical physics. Superalgebras and their representations, supermodules, provide an algebraic framework for formulating supersymmetry. The study of such objects is sometimes called super linear algebra. Superalgebras also play an important role in related field of supergeometry where they enter into the definitions of graded manifolds, supermanifolds and superschemes.

Formal definition

Let K be a fixed commutative ring. In most applications, K is a field such as **R** or **C**.

A **superalgebra** over K is a K-module A with a direct sum decomposition

$$A = A_0 \oplus A_1$$

together with a bilinear multiplication $A \times A \to A$ such that

$$A_i A_j \subseteq A_{i+j}$$

where the subscripts are read modulo 2.

A **superring**, or $\mathbf{Z}_2$-graded ring, is a superalgebra over the ring of integers **Z**.

The elements of A_i are said to be **homogeneous**. The **parity** of a homogeneous element x, denoted by $|x|$, is 0 or 1 according to whether it is in A_0 or A_1. Elements of parity 0 are said to be **even** and those of parity 1 to be **odd**. If x and y are both homogeneous then so is the product xy and $|xy| = |x| + |y|$.

An **associative superalgebra** is one whose multiplication is associative and a **unital superalgebra** is one with a multiplicative identity element. The identity element in a unital superalgebra is necessarily even. Unless otherwise specified, all superalgebras in this article are assumed to be associative and unital.

A **commutative superalgebra** is one which satisfies a graded version of commutativity. Specifically, A is commutative if

$$yx = (-1)^{|x||y|} xy$$

for all homogeneous elements x and y of A.

Examples

- Any algebra over a commutative ring *K* may be regarded as a purely even superalgebra over *K*; that is, by taking A_1 to be trivial.
- Any **Z** or **N**-graded algebra may be regarded as superalgebra by reading the grading modulo 2. This includes examples such as tensor algebras and polynomial rings over *K*.
- In particular, any exterior algebra over *K* is a superalgebra. The exterior algebra is the standard example of a supercommutative algebra.
- The symmetric polynomials and alternating polynomials together form a superalgebra, being the even and odd parts, respectively. Note that this is a different grading from the grading by degree.
- Clifford algebras are superalgebras. They are generally noncommutative.
- The set of all endomorphisms (both even and odd) of a super vector space forms a superalgebra under composition.
- The set of all square supermatrices with entries in *K* forms a superalgebra denoted by $M_{p|q}(K)$. This algebra may be identified with the algebra of endomorphisms of a free supermodule over *K* of rank $p|q$.
- Lie superalgebras are a graded analog of Lie algebras. Lie superalgebras are nonunital and nonassociative; however, one may construct the analog of a universal enveloping algebra of a Lie superalgebra which is a unital, associative superalgebra.

Further definitions and constructions

A superalgebra is an algebra with a $\mathbb{Z}_2$grading ("even" and "odd" elements) such that (i) the bracket of two generators is always antisymmetric except for two odd elements where it is symmetric and (ii) the Jacobi identities are satisfied.[2]

$$[E_i, \{0_k, 0_b\}] = \{[E_i, 0_k], O_b\} + \{[E_i, O_b], 0_k\}$$
$$[0_k, \{O_b, O_a\}] = [\{O_k, O_k\}, O_a] + [\{0_k, 0_a\}, 0_b]$$

The first of these three identities says that the 0 form a representation of the ordinary Lie algebra spanned by E (Consider the 0 as vectors on which the E act.) The second is equivalent to the first if the Killing form is nonsingular. The last identity restricts the possible representations 0 of the ordinary Lie algebra. This relation is the reason that not every ordinary Lie algebra can be extended to a superalgebra.

Even subalgebra

Let *A* be a superalgebra over a commutative ring *K*. The submodule A_0, consisting of all even elements, is closed under multiplication and contains the identity of *A* and therefore forms a subalgebra of *A*, naturally called the **even subalgebra**. It forms an ordinary algebra over *K*.

The set of all odd elements A_1 is an A_0-bimodule whose scalar multiplication is just multiplication in *A*. The product in *A* equips A_1 with a bilinear form

$$\mu : A_1 \otimes_{A_0} A_1 \to A_0$$

such that

$$\mu(x \otimes y) \cdot z = x \cdot \mu(y \otimes z)$$

for all *x*, *y*, and *z* in A_1. This follows from the associativity of the product in *A*.

Grade involution

There is a canonical involutive automorphism on any superalgebra called the **grade involution**. It is given on homogeneous elements by

$$\hat{x} = (-1)^{|x|}x$$

and on arbitrary elements by

$$\hat{x} = x_0 - x_1$$

where x_i are the homogeneous parts of x. If A has no 2-torsion (in particular, if 2 is invertible) then the grade involution can be used to distinguish the even and odd parts of A:

$$A_i = \{x \in A : \hat{x} = (-1)^i x\}.$$

Supercommutativity

The **supercommutator** on A is the binary operator given by

$$[x, y] = xy - (-1)^{|x||y|}yx$$

on homogeneous elements. This can be extended to all of A by linearity. Elements x and y of A are said to **supercommute** if $[x, y] = 0$.

The **supercenter** of A is the set of all elements of A which supercommute with all elements of A:

$$Z(A) = \{a \in A : [a, x] = 0 \text{ for all } x \in A\}.$$

The supercenter of A is, in general, different than the center of A as an ungraded algebra. A commutative superalgebra is one whose supercenter is all of A.

Super tensor product

The graded tensor product of two superalgebras may be regarded as a superalgebra with a multiplication rule determined by:

$$(a_1 \otimes b_1)(a_2 \otimes b_2) = (-1)^{|b_1||a_2|}(a_1 a_2 \otimes b_1 b_2).$$

Generalizations and categorical definition

One can easily generalize the definition of superalgebras to include superalgebras over a commutative superring. The definition given above is then a specialization to the case where the base ring is purely even.

Let R be a commutative superring. A **superalgebra** over R is a R-supermodule A with a R-bilinear multiplication $A \times A \to A$ that respects the grading. Bilinearity here means that

$$r \cdot (xy) = (r \cdot x)y = (-1)^{|r||x|}x(r \cdot y)$$

for all homogeneous elements $r \in R$ and $x, y \in A$.

Equivalently, one may define a superalgebra over R as a superring A together with an superring homomorphism $R \to A$ whose image lies in the supercenter of A.

One may also define superalgebras categorically. The category of all R-supermodules forms a monoidal category under the super tensor product with R serving as the unit object. An associative, unital superalgebra over R can then be defined as a monoid in the category of R-supermodules. That is, a superalgebra is an R-supermodule A with two (even) morphisms

$$\mu : A \otimes A \to A$$
$$\eta : R \to A$$

for which the usual diagrams commute.

Notes

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Algebroid

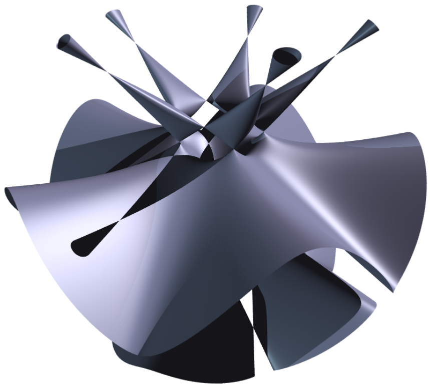
In mathematics, **algebroid** may mean

- algebroid branch, a formal power series branch of an algebraic curve
- algebroid multifunction
- Lie algebroid in the theory of Lie groupoids
- algebroid cohomology

Algebraic Geometry

Algebraic geometry

Algebraic geometry is a branch of mathematics which combines techniques of abstract algebra, especially commutative algebra, with the language and the problems of geometry. It occupies a central place in modern mathematics and has multiple conceptual connections with such diverse fields as complex analysis, topology and number theory. Initially a study of systems of polynomial equations in several variables, the subject of algebraic geometry starts where equation solving leaves off, and it becomes even more important to understand the intrinsic properties of the totality of solutions of a system of equations, than to find some solution; this leads into some of the deepest waters in the whole of mathematics, both conceptually and in terms of technique.



This Togliatti surface is an algebraic surface of degree five.

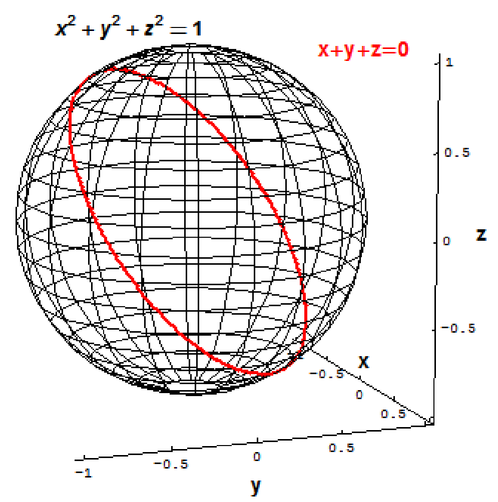
The fundamental objects of study in algebraic geometry are **algebraic varieties**, geometric manifestations of solutions of systems of polynomial equations. Plane algebraic curves, which include lines, circles, parabolas, lemniscates, and Cassini ovals, form one of the best studied classes of algebraic varieties. A point of the plane belongs to an algebraic curve if its coordinates satisfy a given polynomial equation. Basic questions involve relative position of different curves and relations between the curves given by different equations.

Descartes's idea of coordinates is central to algebraic geometry, but it has undergone a series of remarkable transformations beginning in the early 19th century. Before then, the coordinates were assumed to be tuples of real numbers, but this changed when first complex numbers, and then elements of an arbitrary field became acceptable. Homogeneous coordinates of projective geometry offered an extension of the notion of coordinate system in a different direction, and enriched the scope of algebraic geometry. Much of the development of algebraic geometry in the 20th century occurred within an abstract algebraic framework, with increasing emphasis being placed on 'intrinsic' properties of algebraic varieties not dependent on any particular way of embedding the variety in an ambient coordinate space; this parallels developments in topology and complex geometry.

One key distinction between classical projective geometry of 19th century and modern algebraic geometry, in the form given to it by Grothendieck and Serre, is that the former is concerned with the more geometric notion of a point, while the latter emphasizes the more analytic concepts of a regular function and a regular map and extensively draws on sheaf theory. Another important difference lies in the scope of the subject. Grothendieck's idea of **scheme** provides the language and the tools for geometric treatment of arbitrary commutative rings and, in particular, bridges algebraic geometry with algebraic number theory. Andrew Wiles's celebrated proof of Fermat's last theorem is a vivid testament to the power of this approach. André Weil, Grothendieck, and Deligne also demonstrated that the fundamental ideas of topology of manifolds have deep analogues in algebraic geometry over finite fields.

Zeros of simultaneous polynomials

In classical algebraic geometry, the main objects of interest are the vanishing sets of collections of polynomials, meaning the set of all points that simultaneously satisfy one or more polynomial equations. For instance, the two-dimensional sphere in three-dimensional Euclidean space $\mathbf{R}^3$ could be defined as the set of all points (x,y,z) with



Sphere and slanted circle

$$x^2 + y^2 + z^2 - 1 = 0.$$

A "slanted" circle in $\mathbf{R}^3$ can be defined as the set of all points (x,y,z) which satisfy the two polynomial equations

$$x^2 + y^2 + z^2 - 1 = 0,$$
$$x + y + z = 0.$$

Affine varieties

First we start with a field k. In classical algebraic geometry, this field was always the complex numbers $\mathbf{C}$, but many of the same results are true if we assume only that k is algebraically closed. We define $\mathbf{A}^n(k)$ (or more simply $\mathbf{A}^n$, when k is clear from the context), called the **affine n-space over k**, to be k^n. The purpose of this apparently superfluous notation is to emphasize that one 'forgets' the vector space structure that k^n carries. Abstractly speaking, $\mathbf{A}^n$ is, for the moment, just a collection of points.

A function $f : \mathbf{A}^n \to \mathbf{A}^1$ is said to be **regular** if it can be written as a polynomial, that is, if there is a polynomial p in $k[x_1,...,x_n]$ such that $f(t_1,...,t_n) = p(t_1,...,t_n)$ for every point $(t_1,...,t_n)$ of $\mathbf{A}^n$.

Regular functions on affine n-space are thus exactly the same as polynomials over k in n variables. We will refer to the set of all regular functions on $\mathbf{A}^n$ as $k[\mathbf{A}^n]$.

We say that a polynomial *vanishes* at a point if evaluating it at that point gives zero. Let S be a set of polynomials in $k[\mathbf{A}^n]$. The *vanishing set of S* (or *vanishing locus*) is the set $V(S)$ of all points in $\mathbf{A}^n$ where every polynomial in S vanishes. In other words,

$$V(S) = \{(t_1, \ldots, t_n) | \forall p \in S, p(t_1, \ldots, t_n) = 0\}.$$

A subset of $\mathbf{A}^n$ which is $V(S)$, for some S, is called an **algebraic set**. The V stands for *variety* (a specific type of algebraic set to be defined below).

Given a subset U of $\mathbf{A}^n$, can one recover the set of polynomials which generate it? If U is *any* subset of $\mathbf{A}^n$, define $I(U)$ to be the set of all polynomials whose vanishing set contains U. The I stands for ideal: if two polynomials f and g both vanish on U, then $f+g$ vanishes on U, and if h is any polynomial, then hf vanishes on U, so $I(U)$ is always an ideal of $k[\mathbf{A}^n]$.

Two natural questions to ask are:

- Given a subset U of $\mathbf{A}^n$, when is $U = V(I(U))$?

- Given a set S of polynomials, when is $S = I(V(S))$?

The answer to the first question is provided by introducing the Zariski topology, a topology on $\mathbf{A}^n$ which directly reflects the algebraic structure of $k[\mathbf{A}^n]$. Then $U = V(I(U))$ if and only if U is a Zariski-closed set. The answer to the second question is given by Hilbert's Nullstellensatz. In one of its forms, it says that $I(V(S))$ is the prime radical of the ideal generated by S. In more abstract language, there is a Galois connection, giving rise to two closure operators; they can be identified, and naturally play a basic role in the theory; the example is elaborated at Galois connection.

For various reasons we may not always want to work with the entire ideal corresponding to an algebraic set U. Hilbert's basis theorem implies that ideals in $k[\mathbf{A}^n]$ are always finitely generated.

An algebraic set is called **irreducible** if it cannot be written as the union of two smaller algebraic sets. An irreducible algebraic set is also called a **variety**. It turns out that an algebraic set is a variety if and only if the polynomials defining it generate a prime ideal of the polynomial ring.

Regular functions

Just as continuous functions are the natural maps on topological spaces and smooth functions are the natural maps on differentiable manifolds, there is a natural class of functions on an algebraic set, called regular functions. A **regular function** on an algebraic set V contained in $\mathbf{A}^n$ is defined to be the restriction of a regular function on $\mathbf{A}^n$, in the sense we defined above.

It may seem unnaturally restrictive to require that a regular function always extend to the ambient space, but it is very similar to the situation in a normal topological space, where the Tietze extension theorem guarantees that a continuous function on a closed subset always extends to the ambient topological space.

Just as with the regular functions on affine space, the regular functions on V form a ring, which we denote by $k[V]$. This ring is called the **coordinate ring of *V***.

Since regular functions on V come from regular functions on $\mathbf{A}^n$, there should be a relationship between their coordinate rings. Specifically, to get a function in $k[V]$ we took a function in $k[\mathbf{A}^n]$, and we said that it was the same as another function if they gave the same values when evaluated on V. This is the same as saying that their difference is zero on V. From this we can see that $k[V]$ is the quotient $k[\mathbf{A}^n]/\mathrm{I}(V)$.

The category of affine varieties

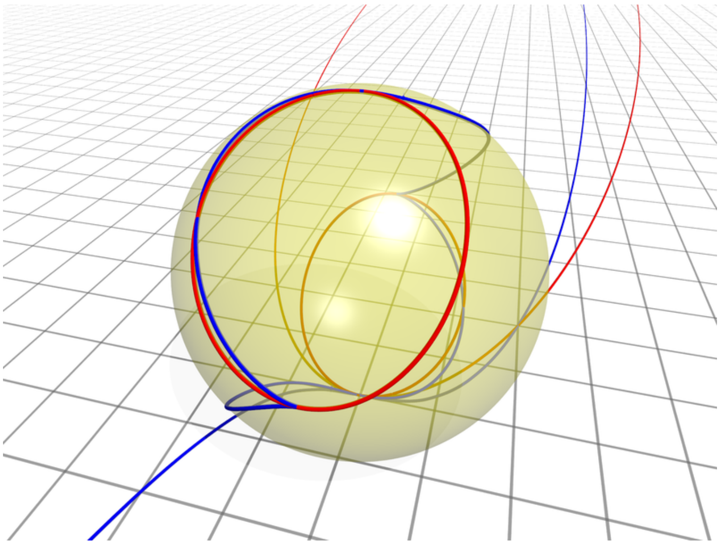
Using regular functions from an affine variety to $\mathbf{A}^1$, we can define regular functions from one affine variety to another. First we will define a regular function from a variety into affine space: Let V be a variety contained in $\mathbf{A}^n$. Choose m regular functions on V, and call them $f_1, ..., f_m$. We define a **regular function** f from V to $\mathbf{A}^m$ by letting $f(t_1, ..., t_n) = (f_1, ..., f_m)$. In other words, each f_i determines one coordinate of the range of f.

If V' is a variety contained in $\mathbf{A}^m$, we say that f is a **regular function** from V to V' if the range of f is contained in V'.

This makes the collection of all affine varieties into a category, where the objects are affine varieties and the morphisms are regular maps. The following theorem characterizes the category of affine varieties:

> The category of affine varieties is the opposite category to the category of finitely generated integral k-algebras and their homomorphisms.

Projective space


parabola ($y = x^2$, red) and cubic ($y = x^3$, blue) in projective space

Consider the variety $V(y - x^2)$. If we draw it, we get a parabola. As x increases, the slope of the line from the origin to the point (x, x^2) becomes larger and larger. As x decreases, the slope of the same line becomes smaller and smaller.

Compare this to the variety $V(y - x^3)$. This is a cubic equation. As x increases, the slope of the line from the origin to the point (x, x^3) becomes larger and larger just as before. But unlike before, as x decreases, the slope of the same line again becomes larger and larger. So the behavior "at infinity" of $V(y - x^3)$ is different from the behavior "at infinity" of $V(y - x^2)$. It is, however, difficult to make the concept of "at infinity" meaningful, if we restrict to working in affine space.

The remedy to this is to work in projective space. Projective space has properties analogous to those of a compact Hausdorff space. Among other things, it lets us make precise the notion of "at infinity" by including extra points. The behavior of a variety at those extra points then gives us more information about it. As it turns out, $V(y - x^3)$ has a singularity at one of those extra points, but $V(y - x^2)$ is smooth.

While projective geometry was originally established on a synthetic foundation, the use of homogeneous coordinates allowed the introduction of algebraic techniques. Furthermore, the introduction of projective techniques made many theorems in algebraic geometry simpler and sharper: For example, Bézout's theorem on the number of intersection points between two varieties can be stated in its sharpest form only in projective space. For this reason, projective space plays a fundamental role in algebraic geometry.

The modern viewpoint

The modern approaches to algebraic geometry redefine and effectively extend the range of basic objects in various levels of generality to schemes, formal schemes, ind-schemes, algebraic spaces, algebraic stacks, derived algebraic stacks and so on. The need for this arises already from the useful ideas within theory of varieties, e.g. the formal functions of Zariski can be accommodated by introducing nilpotent elements in structure rings; considering spaces of loops and arcs, constructing quotients by group actions and developing formal grounds for natural intersection theory and deformation theory lead to some of the further extensions.

Most remarkably, in late 1950-s, algebraic varieties are subsumed in Alexander Grothendieck's concept of a scheme. Their local objects are affine schemes or prime spectra which are locally ringed spaces which form a category which is antiequivalent to the category of commutative unital rings, extending the duality between the category of affine algebraic varieties over a field k, and the category of finitely generated reduced k-algebras. The gluing is along Zariski topology; one can glue within the category of locally ringed spaces, but also, using the Yoneda embedding, within the more abstract category of presheaves of sets over the category of affine schemes. The Zariski topology in the set theoretic sense is then replaced by a Zariski topology in the sense of Grothendieck topology. Grothendieck introduced Grothendieck topologies having in mind more exotic but geometrically finer and more sensitive examples than the crude Zariski topology, namely the etale topology, and the two flat Grothendieck topologies: ffpf and fpqc; nowadays some other examples became prominent including Nisnevich topology. Sheaves can be furthermore generalized to stacks in the sense of Grothendieck, usually with some additional representability conditions leading to Artin stacks and, even finer, Deligne-Mumford stacks, both often called algebraic stacks.

Sometimes other algebraic sites replace the category of affine schemes. For example, Nikolai Durov has introduced commutative algebraic monads as a generalization of local objects in a generalized algebraic geometry. Versions of a tropical geometry, of an absolute geometry over a field of one element and an algebraic analogue of Arakelov's geometry were realized in this setup.

Another formal generalization is possible to Universal algebraic geometry in which every variety of algebra has its own algebraic geometry. The term *variety of algebra* should not be confused with *algebraic variety*.

The language of schemes, stacks and generalizations has proved to be a valuable way of dealing with geometric concepts and became cornerstones of modern algebraic geometry.

Derived algebraic geometry

Algebraic stacks can be further generalized and for many practical questions like deformation theory and intersection theory, this is often the most natural approach. One can extend the Grothendieck site of affine schemes to a higher categorical site of derived affine schemes, by replacing the commutative rings with an infinity category of differential graded commutative algebras, or of simplicial commutative rings or a similar category with an appropriate variant of a Grothendieck topology. One can also replace presheaves of sets by presheaves of simplicial sets (or of infinity groupoids). Then, in presence of an appropriate homotopic machinery one can develop a notion of derived stack as such a presheaf on the infinity category of derived affine schemes, which is satifsying certain infinite categorical version of a sheaf axiom (and to be algebraic, inductively a sequence of representability conditions). Quillen model categories, Segal categories and quasicategories are some of the most often used tools to formalize this yielding the **derived algebraic geometry**, introduced by the school of Carlos Simpson, including Andre Hirschowitz, Bertrand Toën, Gabrielle Vezzosi, Michel Vaquié and others; and recently systematized and applied by Jacob Lurie. Another (noncommutative) version of derived algebraic geometry, using A-infinity categories has been developed from early 1990-s by Maxim Kontsevich and followers.

History

Prehistory: Before the 19th century

Some of the roots of algebraic geometry date back to the work of the Hellenistic Greeks from the 5th century BC. The Delian problem, for instance, was to construct a length x so that the cube of side x contained the same volume as the rectangular box a^2b for given sides a and b. Menechmus (circa 350 BC) considered the problem geometrically by intersecting the pair of plane conics $ay = x^2$ and $xy = ab$.[1] The later work, in the 3rd century BC, of Archimedes and Apollonius studied more systematically problems on conic sections,[2] and also involved the use of coordinates.[1] The Arab mathematicians were able to solve by purely algebraic means certain cubic equations, and then to interpret the results geometrically. This was done, for instance, by Ibn al-Haytham in the 10th century AD.[3] Subsequently, Persian mathematician Omar Khayyám (born 1048 A.D.) discovered the general method of solving cubic equations by intersecting a parabola with a circle.[4] Each of these early developments in algebraic geometry dealt with questions of finding and describing the intersections of algebraic curves.

Such techniques of applying geometrical constructions to algebraic problems were also adopted by a number of Renaissance mathematicians such as Gerolamo Cardano and Niccolò Fontana "Tartaglia" on their studies of the cubic equation. The geometrical approach to construction problems, rather than the algebraic one, was favored by most 16th and 17th century mathematicians, notably Blaise Pascal who argued against the use of algebraic and analytical methods in geometry.[5] The French mathematicians Franciscus Vieta and later René Descartes and Pierre de Fermat revolutionized the conventional way of thinking about construction problems through the introduction of coordinate geometry. They were interested primarily in the properties of *algebraic curves*, such as those defined by

Diophantine equations (in the case of Fermat), and the algebraic reformulation of the classical Greek works on conics and cubics (in the case of Descartes).

During the same period, Blaise Pascal and Gérard Desargues approached geometry from a different perspective, developing the synthetic notions of projective geometry. Pascal and Desargues also studied curves, but from the purely geometrical point of view: the analog of the Greek *ruler and compass construction*. Ultimately, the analytic geometry of Descartes and Fermat won out, for it supplied the 18th century mathematicians with concrete quantitative tools needed to study physical problems using the new calculus of Newton and Leibniz. However, by the end of the 18th century, most of the algebraic character of coordinate geometry was subsumed by the *calculus of infinitesimals* of Lagrange and Euler.

Nineteenth and early 20th century

It took the simultaneous 19th century developments of non-Euclidean geometry and Abelian integrals in order to bring the old algebraic ideas back into the geometrical fold. The first of these new developments was seized up by Edmond Laguerre and Arthur Cayley, who attempted to ascertain the generalized metric properties of projective space. Cayley introduced the idea of *homogeneous polynomial forms*, and more specifically quadratic forms, on projective space. Subsequently, Felix Klein studied projective geometry (along with other sorts of geometry) from the viewpoint that the geometry on a space is encoded in a certain class of transformations on the space. By the end of the 19th century, projective geometers were studying more general kinds of transformations on figures in projective space. Rather than the projective linear transformations which were normally regarded as giving the fundamental Kleinian geometry on projective space, they concerned themselves also with the higher degree birational transformations. This weaker notion of congruence would later lead members of the 20th century Italian school of algebraic geometry to classify algebraic surfaces up to birational isomorphism.

The second early 19th century development, that of Abelian integrals, would lead Bernhard Riemann to the development of Riemann surfaces.

Twentieth century

B. L. van der Waerden, Oscar Zariski, André Weil and others attempted to develop a rigorous foundation for algebraic geometry based on contemporary commutative algebra, including valuation theory and the theory of ideals.

In the 1950s and 1960s Jean-Pierre Serre and Alexander Grothendieck recast the foundations making use of sheaf theory. Later, from about 1960, and largely spearheaded by Grothendieck, the idea of schemes was worked out, in conjunction with a very refined apparatus of homological techniques. After a decade of rapid development the field stabilized in the 1970s, and new applications were made, both to number theory and to more classical geometric questions on algebraic varieties, singularities and moduli.

An important class of varieties, not easily understood directly from their defining equations, are the abelian varieties, which are the projective varieties whose points form an abelian group. The prototypical examples are the elliptic curves, which have a rich theory. They were instrumental in the proof of Fermat's last theorem and are also used in elliptic curve cryptography.

While much of algebraic geometry is concerned with abstract and general statements about varieties, methods for effective computation with concretely-given polynomials have also been developed. The most important is the technique of Gröbner bases which is employed in all computer algebra systems. Based on these methods, several solvers may compute all the solutions of a system of polynomial equations whose associated variety has dimension zero and thus consists in a finite number of points.

Applications

Algebraic geometry now finds application in statistics,[6] control theory,[7] robotics,[8] error-correcting codes,[9] phylogenetics[10] and geometric modelling.[11] There are also connections to string theory,[12] game theory,[13] graph matchings,[14] solitons[15] and integer programming.[16] Google scholar lists hundreds of more studies on algebraic geometry in biology [17], chemistry [18], economics [19], physics [20] and of course other areas of mathematics [21].

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List of algebraic geometry topics

This is a **list of algebraic geometry topics**, by Wikipedia page.

Classical topics in projective geometry

- Affine space
- Projective space
- Projective line, cross-ratio
- Projective plane
  - Line at infinity
  - Complex projective plane
- Complex projective space
- Plane at infinity, hyperplane at infinity
- Projective frame
- Projective transformation
- Fundamental theorem of projective geometry
- Duality (projective geometry)
- Real projective plane
- Real projective space
- Segre embedding, multi-way projective space
- Rational normal curve

Algebraic curves

- Conics, Pascal's theorem, Brianchon's theorem
- Twisted cubic
- Elliptic curve, cubic curve
  - Elliptic function, Jacobi's elliptic functions, Weierstrass's elliptic functions
  - Elliptic integral
  - Complex multiplication
  - Weil pairing
- Hyperelliptic curve
- Klein quartic
- modular curve
  - modular equation
  - modular function
  - modular group
  - Supersingular primes
- Fermat curve
- Bézout's theorem
- Brill–Noether theory
- Edwards curve
- Genus (mathematics)
- Riemann surface
- Riemann–Hurwitz formula
- Riemann–Roch theorem

- Abelian integral
- Differential of the first kind
- Jacobian variety
  - Generalized Jacobian
- Hurwitz's automorphisms theorem
- Clifford's theorem
- Gonality of an algebraic curve
- Weil's reciprocity law
- Goppa code

Algebraic surfaces

- Enriques-Kodaira classification
- List of algebraic surfaces
- Ruled surface
- Cubic surface
- Veronese surface
- Del Pezzo surface
- Rational surface
- Enriques surface
- K3 surface
- Hodge index theorem
- Elliptic surface
- Surface of general type
- Zariski surface

Algebraic geometry: classical approach

- Algebraic variety
  - Hypersurface
  - Quadric
  - Dimension of an algebraic variety
  - Hilbert's Nullstellensatz
  - Complete variety
  - Elimination theory
  - Quasiprojective variety
    - Gröbner basis
  - Canonical bundle
  - Complete intersection
  - Serre duality
  - Arithmetic genus, geometric genus, irregularity
- Tangent space, Zariski tangent space
- Function field
- Ample vector bundle
- Linear system of divisors
- Birational geometry
  - Blowing up

  - Rational variety
  - Unirational variety
- Intersection number
  - Intersection theory
  - Serre's multiplicity conjectures
- Albanese variety
- Picard group
- Pluricanonical ring
- Modular form
- Moduli space
- Modular equation
  - J-invariant
- Algebraic function
- Algebraic form
- Addition theorem
- Invariant theory
  - Symbolic method of invariant theory
- Geometric invariant theory
- Toric geometry
- Deformation theory
- Singular point, non-singular
- Singularity theory
  - Newton polygon
- Weil conjectures

Complex manifolds

- Kähler manifold
- Calabi–Yau manifold
- Stein manifold
- Hodge theory
- Hodge cycle
- Hodge conjecture
- Algebraic geometry and analytic geometry
- Mirror symmetry

Algebraic groups

- Identity component
- Linear algebraic group
  - Additive group
  - Multiplicative group
  - Borel subgroup
  - Parabolic subgroup
  - Radical of an algebraic group
  - Unipotent radical
  - Lie-Kolchin theorem

- Mumford conjecture
- Abelian variety
  - Theta function
- Grassmannian
- Flag manifold
- Algebraic torus
- Weil restriction
- Differential Galois theory

Contemporary foundations

Main article glossary of scheme theory

Commutative algebra

- Prime ideal
- Valuation (mathematics)
- Regular local ring
- Regular sequence (algebra)
- Cohen–Macaulay ring
- Gorenstein ring
- Koszul complex
- Spectrum of a ring
- Zariski topology
- Kähler differential
- Generic flatness
- Irrelevant ideal

Sheaf theory

- Locally ringed space
- Coherent sheaf
- Invertible sheaf
- Sheaf cohomology
- Hirzebruch–Riemann–Roch theorem
- Grothendieck–Riemann–Roch theorem
- Coherent duality
- Dévissage

Schemes

- Affine scheme
- Scheme
- Glossary of scheme theory
- Éléments de géométrie algébrique
- Grothendieck's Séminaire de géométrie algébrique
- Flat morphism
- Finite morphism
- Quasi-finite morphism
- Group scheme

- Semistable elliptic curve
- Grothendieck's relative point of view

Category theory

- Grothendieck topology
- Topos
- Descent (category theory)
  - Grothendieck's Galois theory
- Algebraic stack
- Gerbe
- Etale cohomology
- Motive (mathematics)
- Motivic cohomology
- Homotopical algebra

Algebraic geometers

- Niels Henrik Abel
- Carl Gustav Jakob Jacobi
- Jakob Steiner
- Julius Plücker
- Bernhard Riemann
- William Kingdon Clifford
- Italian school of algebraic geometry
  - Guido Castelnuovo
  - Francesco Severi
- Solomon Lefschetz
- Oscar Zariski
- Erich Kähler
- W. V. D. Hodge
- Kunihiko Kodaira
- André Weil
- Jean-Pierre Serre
- Alexander Grothendieck
- David Mumford
- Igor Shafarevich
- Heisuke Hironaka
- Shigefumi Mori
- Vladimir Voevodsky

Duality (projective geometry)

In the geometry of the projective plane, **duality** refers to geometric transformations that replace points by lines and lines by points while preserving incidence properties among the transformed objects. The existence of such transformations leads to a general principle, that any theorem about incidences between points and lines in the projective plane may be transformed into another theorem about lines and points, by a substitution of the appropriate words.

Duality in the projective plane is a special case of duality for projective spaces, transformations that interchange

dimension + codimension.

That is, in a projective space of dimension *n*, the points (dimension 0) are made to correspond with hyperplanes (codimension 1), the lines joining two points (dimension 1) are made to correspond with the intersection of two hyperplanes (codimension 2), and so on.

Duality in terms of vector space

The points of *n*-dimensional projective space over a field **F**, called $\mathbf{CF}^n$, can be taken to be the nonzero vectors in the (n + 1)-dimensional vector space over **F**, where we identify two vectors which differ by a scalar factor. Another way to put it is that the points of *n*-dimensional projective space are the lines through the origin in $\mathbf{F}^{n+1}$. Also the *n*-dimensional subspaces of $\mathbf{F}^{n+1}$ represent the (n – 1)-dimensional hyperplanes of projective *n*-space over **F**.

A nonzero vector **u** in $\mathbf{F}^{n+1}$ also determines an n-dimensional subspace, by means of the equation

$$u_0x_0 + \cdots + u_nx_n = 0$$

where u_i is the ith coordinate of **u**, starting from zero, all u_i's are from field **F**. In terms of the dot product we can write this as $u \cdot x = 0$, where **u** is the vector representing a hyperplane, and **x** the vector representing a point. The dot product is symmetrical, and the same vector **u** represents both a point and a hyperplane. Hence we have a duality between points and hyperplanes, which extends to a duality between the line generated by two points and the intersection of two hyperplanes, and so forth.

Points and lines in the plane

Notice that both points and lines can be represented (on a plane) by means of ordered pairs. A point is represented by the ordered pair *(x,y)*, where *x* is the abscissa and *y* is the ordinate, which together are coordinates of the point. A line can likewise be represented by an ordered pair *(m,b)* where *m* is the slope and *b* is the y-intercept.

Given three points

$$P_1 : (x_1, y_1),$$
$$P_2 : (x_2, y_2),$$
$$P_3 : (x_3, y_3);$$

these three points are collinear (i.e., they lie on the same line) if and only if their coordinates satisfy the equation

$$\frac{y_2 - y_1}{x_2 - x_1} = \frac{y_3 - y_2}{x_3 - x_2} = \frac{y_3 - y_1}{x_3 - x_1} \qquad (1).$$

Likewise, given three lines

$$L_1 : (m_1, b_1),$$
$$L_2 : (m_2, b_2),$$
$$L_3 : (m_3, b_3);$$

one can verify that these three lines are concurrent (i.e., all share an intersection) if and only if their parameters satisfy the equation

$$\frac{b_2 - b_1}{m_2 - m_1} = \frac{b_3 - b_2}{m_3 - m_2} = \frac{b_3 - b_1}{m_3 - m_1}. \qquad (2)$$

Equations (1) and (2) are equivalent to each other up to an exchange of *x* with *m* and *y* with *b*. Therefore there exists a way to exchange lines with points in such a way that concurrency is exchanged with collinearity.

It is possible to distinguish lines from points by conjugating ordered pairs. That is, let line *(m,b)* be represented instead by its conjugate $(m, -b)^\star$. Then it can be verified that the intersection $L_1.L_2$ of a pair of lines L_1 and L_2 is

$$(m_1, b_1)^\star.(m_2, b_2)^\star = \left(\frac{b_2 - b_1}{m_2 - m_1}, \frac{m_1 b_2 - m_2 b_1}{m_2 - m_1}\right) \qquad (3)$$

where b_1 and b_2 are negative y-intercepts. Also, the common line $P_1.P_2$ passing through a pair of points P_1 and P_2 is

$$(x_1, y_1).(x_2, y_2) = \left(\frac{y_2 - y_1}{x_2 - x_1}, \frac{x_1 y_2 - x_2 y_1}{x_2 - x_1}\right)^\star. \qquad (4)$$

Equation (4) can be seen to be the same as equation (3), after exchanging *m* with *x* and *b* with *y*, and applying the following rules of conjugation:

$$A^{\star\star} = A, \qquad (5)$$

$$A = B \to A^\star = B^\star, \qquad (6)$$

$$(A.B)^\star = A^\star.B^\star = B^\star.A^\star. \qquad (7)$$

Indeed, if equation (3) is represented as

$$A^\star.B^\star = C$$

then applying rule (6) yields

$$(A^\star.B^\star)^\star = C^\star.$$

Applying rule (7) then yields

$$A^{\star\star}.B^{\star\star} = C^\star$$

and applying rule (5) finally yields

$$A.B = C^\star,$$

which is equation (4).

Thus it is possible to imagine a pair of planes S_1 and S_2, and a bijective relation between loci of points in the two planes, such that points in S_2 correspond to lines in S_1, and points in S_1 correspond to lines in S_2.

Great circles

One way to establish such bijection is to model the real projective plane, not as an extended affine plane, but as a "unit sphere modulo antipodes", *i.e.* a unit sphere in which antipodal points are equivalent. Then through points P_1 and P_2 in S_1 passes a geodesic line L_3 which is actually a great circle. But to these two original points correspond a pair of great circles L_1 and L_2 in S_2, such that if S_2 and S_1 are superposed, then L_1 is the unique great circle perpendicular to the line through the pair of points P_1 *1, and L_2 is the unique great circle perpendicular to the line through the pair of points P_2. These great circles L_1 and L_2 intersect at a pair of points P_3 in S_2. The vector through P_3 is the cross product of the vectors through P_1 and P_2. Then the unique great circle perpendicular to the line passing through the pair of points P_3 is geodesic line L_3 in S_1.

Therefore to every great circle in S_1 corresponds a unique pair of points (which are actually the same point) in S_2, such that if S_1 and S_2 are superposed, then the (3-D) line passing through the pair of points is perpendicular to (the plane in 3-D of) the great circle. The above sentence remains true if S_1 and S_2 are exchanged. This establishes the bijective nature of the duality in the projective plane.

It must be noted that in the "unit sphere modulo antipodes", one "geodesic line", *i.e.* great circle, must be chosen to be the line at infinity if the surface is to be mapped to an extended affine plane. This line may be chosen to be the equator by convention.

- *1 When we say a line through pair of points *P*, or simply a line through *P*, we refer to the three dimensional euclidian line that passes through the antipodal points represented by *P*. When we say that a geodesic line, or a great circle, *L* is perpendicular to the line passing through the pair of points *P*, we mean that *L* lies on the plane that is perpendicular to, and intersects at the midpoint of, the straight line segment in euclidian space that connects the antipodal points that is represented by *P*. In other words, *L* is the set of points equidistant in euclidian space to the antipodal points represented by *P*. *L* is unique for *P*.

Three dimensions

There is also a duality in projective 3-space, in which points correspond to planes, and lines correspond to lines. This is analogous to duality of polyhedra in solid geometry, where points are dual to faces, and sides are dual to sides, so that the icosahedron is dual to the dodecahedron, and the cube is dual to the octahedron.

Mapping the sphere onto the plane

The unit sphere modulo −1 model of the projective plane is isomorphic (w.r.t. incidence properties) to the planar model: the affine plane extended with a projective line at infinity.

To map a point on the sphere to a point on the plane, let the plane be tangent to the sphere at some point which shall be the origin of the plane's coordinate system (2-D origin). Then construct a line passing through the center of the sphere (3-D origin) and the point on the sphere. This line intersects the plane at a point which is the projection of the point on the sphere onto the plane (or vice versa).

This projection can be used to define a one-to-one onto mapping

$$f : [0, \pi/2] \times [0, 2\pi] \to \mathbb{R}P^2.$$

If points in $\mathbb{R}P^2$ are expressed in homogeneous coordinates, then

$$f : (\theta, \phi) \mapsto [\cos\phi : \sin\phi : \cot\theta],$$

$$f^{-1} : [x : y : z] \mapsto \left(\arctan\sqrt{\left(\frac{x}{z}\right)^2 + \left(\frac{y}{z}\right)^2}, \operatorname{arctan}_2(y, x)\right).$$

Also, lines in the planar model are projections of great circles of the sphere. This is so because through any line in the plane pass an infinitude of different planes: one of these planes passes through the 3-D origin, but a plane passing through the 3-D origin intersects the sphere along a great circle.

As we have seen, any great circle in the unit sphere has a projective point perpendicular to it, which can be defined as its dual. But this point is a pair of antipodal points on the unit sphere, through both of which passes a unique 3-D line, and this line extended past the unit sphere intersects the tangent plane at a point, which means that there is a geometric way to associate a unique point on the plane to every line on the plane, such that the point is the dual of the line.

Duality mapping defined

Given a line L in the projective plane, what is its dual point? Draw a line L' passing through the 2-D origin and perpendicular to line L. Then pick a point P on line L' on the other side of the origin from line L, such that the distance of point P to the origin is the reciprocal of the distance of line L to the origin.

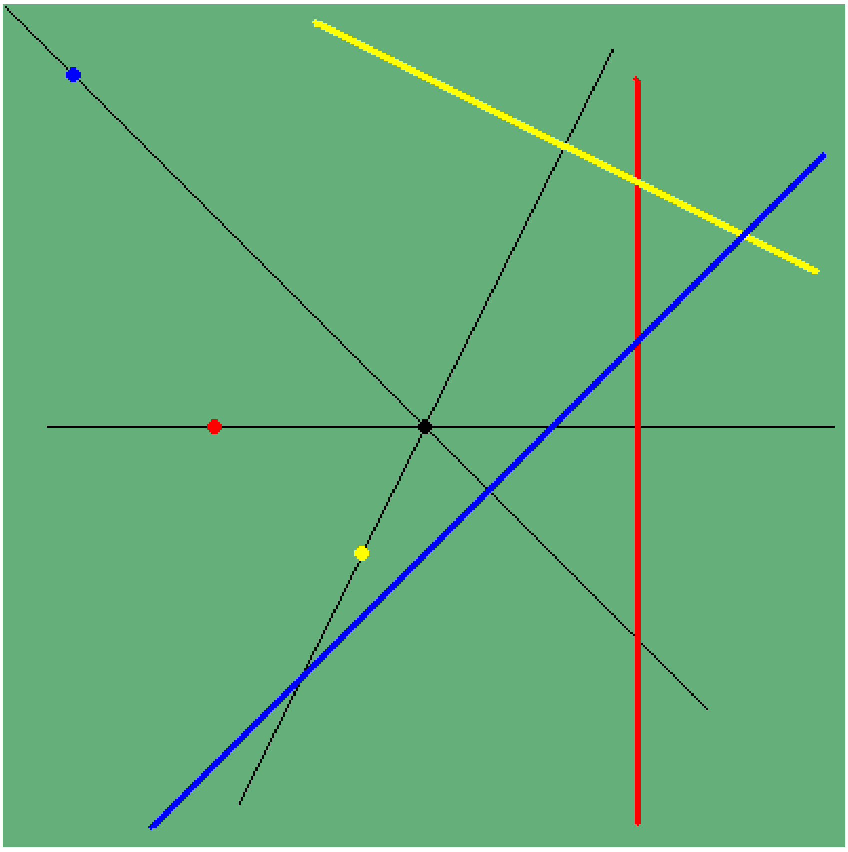

Figure 1. Three pairs of dual points and lines: one red pair, one yellow pair, and one blue pair. The duality is an isomorphism of incidence, so that, e.g., the line passing through the red and yellow points is dual to the intersection of the red and yellow lines.

Expressed algebraically, let g be a one-to-one mapping from the projective plane onto itself:

$$g : \mathbb{R}P^2 \to \mathbb{R}P^2$$

such that

$$g : [m : b : 1]_L \mapsto [m : -1 : b]$$

and

$$g : [x : y : 1] \mapsto [x : 1 : -y]_L$$

where the L subscript is used to semantically distinguish line coordinates from point coordinates. In words, affine line (m, b) with slope m and y-intercept b is the dual of point (m/b, $-1/b$). If b=0 then the line passes through the 2-D origin and its dual is the ideal point [m : −1 : 0].

The affine point with Cartesian coordinates (x,y) has as its dual the line whose slope is $-x/y$ and whose y-intercept is $-1/y$. If the point is the 2-D origin [0:0:1], then its dual is $[0:1:0]_L$ which is the line at infinity. If the point is [x:0:1], on the x-axis, then its dual is line $[x:1:0]_L$ which shall be interpreted as a line whose slope is vertical and whose x-intercept is $-1/x$.

If a point or a line's homogeneous coordinates are represented as a vector in 3x1 matrix form, then the duality mapping g can be represented as a trilinear transformation, a 3x3 matrix

$$G = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}$$

whose inverse is

$$G^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}.$$

Matrix *G* has one real eigenvalue: one, whose eigenvector is [1:0:0]. The line $[1:0:0]_L$ is the y-axis, whose dual is the ideal point [1:0:0] which is the intersection of the ideal line with the x-axis.

Notice that $[1:0:0]_L$ is the y-axis, $[0:1:0]_L$ is the line at infinity, and $[0:0:1]_L$ is the x-axis. In 3-space, matrix *G* is a 90° rotation about the x-axis which turns the y-axis into the z-axis. In projective 2-space, matrix *G* is a projective transformation which maps points to points, lines to lines, conic sections to conic sections: it exchanges the line at infinity with the x-axis and maps the y-axis onto itself through a Möbius transformation. As a duality, matrix *G* pairs up each projective line with its dual projective point.

Preservation of incidence

The duality mapping *g* is an isomorphism with respect to the incidence properties (such as collinearity and concurrency). The mapping *g* has this property: given a pair of lines L_1 and L_2 which intersect at a point *P*, then their dual points gL_1 and gL_2 define the unique line $g^{-1}P$:

$$g^{-1}(L_1 \cap L_2) = gL_1.gL_2.$$

Given points P_1 and P_2 through which passes line *L*, $P_1.P_2 = L$, then what is the intersection of lines $g^{-1}P_1$ and $g^{-1}P_2$? If $g^{-1}P_1 \cap g^{-1}P_2 = P$ then

$$\begin{aligned} g^{-1}P = g^{-1}(g^{-1}P_1 \cap g^{-1}P_2) &= g(g^{-1}P_1).g(g^{-1}P_2) \\ &= P_1.P_2 \\ &= L \end{aligned}$$

so that

$$g(g^{-1}P) = gL$$
$$P = gL$$
$$\therefore\ g(P_1.P_2) = g^{-1}P_1 \cap g^{-1}P_2$$

Given a pair of affine points in homogeneous coordinates, the line passing through them is

$$[x_1 : y_1 : 1].[x_2 : y_2 : 1] = g^{-1}([x_1 : y_1 : 1] \times [x_2 : y_2 : 1])$$

where the cross product is computed just as it would for an ordinary pair vectors in 3-space.

From this last equation can be derived the intersection of lines, by using the mapping *g* to "plug in" the lines into the slots for points:

$$g[m_1 : b_1 : 1]_L.g[m_2 : b_2 : 1]_L = g^{-1}(g[m_1 : b_1 : 1]_L \times g[m_2 : b_2 : 1]_L)$$
$$g(g[m_1 : b_1 : 1]_L.g[m_2 : b_2 : 1]_L) = g[m_1 : b_1 : 1]_L \times g[m_2 : b_2 : 1]_L$$
$$[m_1 : b_1 : 1]_L \cap [m_2 : b_2 : 1]_L = g([m_1 : b_1 : 1]_L \times [m_2 : b_2 : 1]_L)$$

where mapping *g* is seen to distribute with respect to the cross product: i.e. *g* is an isomorphism of cross product.

Theorem. The duality mapping *g* is an isomorphism of cross product. I.e. *g* is distributive w.r.t. cross product.

Proof. Given points *A*=(*a*:*b*:*c*) and *B*=(*d*:*e*:*f*), their cross product is $(a : b : c) \times (d : e : f) = (bf - ce : cd - af : ae - bd)$ but

$$g(a : b : c) = (a : -c : b),$$

$$g(d:e:f) = (d:-f:e),$$
$$(a:-c:b) \times (d:-f:e) = (-ce+bf:bd-ae:-af+cd)$$
$$= g(bf-ce:cd-af:ae-bd).$$

Therefore

$$g(A \times B) = gA \times gB\ .$$

Q.E.D.

Combinatorial duality

If one defines a projective plane axiomatically as an incidence structure, in terms of a set *P* of points, a set *L* of lines, and an incidence relation *I* that determines which points lie on which lines, then one may define duality abstractly thus.

If we interchange the role of "points" and "lines" in

$$C = (P, L, I),$$

the dual structure

$$C^* = (L, P, I^*)$$

is obtained, where I^* is the inverse relation of *I*.

References

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References

[1] http://mathworld.wolfram.com/DualityPrinciple.html

Universal algebraic geometry

In **universal algebraic geometry**, algebraic geometry is generalized from the geometry of rings to geometry of arbitrary varieties of algebras, so that every *variety of algebras* has its own algebraic geometry. Note that the two terms algebraic variety and *variety of algebras* should not be confused.

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- Seven Lectures on the Universal Algebraic Geometry [1]

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[1] http://arxiv.org/abs/math/0204245

Motive (algebraic geometry)

In algebraic geometry, a **motive** (or sometimes **motif**, following French usage) denotes 'some essential part of an algebraic variety'. To date, pure motives have been defined, while conjectural mixed motives have not. Pure motives are triples (X, p, m), where X is a smooth projective variety, $p : X \vdash X$ is an idempotent correspondence, and m an integer. A morphism from (X, p, m) to (Y, q, n) is given by a correspondence of degree $n - m$.

As far as mixed motives, following Alexander Grothendieck, mathematicians are working to find a suitable definition which will then provide a "universal" cohomology theory. In terms of category theory, it was intended to have a definition via splitting idempotents in a category of algebraic correspondences. The way ahead for that definition has been blocked for some decades, by the failure to prove the standard conjectures on algebraic cycles. This prevents the category from having 'enough' morphisms, as can currently be shown. While the category of motives was supposed to be the **universal Weil cohomology** much discussed in the years 1960-1970, that hope for it remains unfulfilled. On the other hand, by a quite different route, motivic cohomology now has a technically-adequate definition.

Introduction

The theory of motives was originally conjectured as an attempt to unify a rapidly multiplying array of cohomology theories, including Betti cohomology, de Rham cohomology, *l*-adic cohomology, and crystalline cohomology. The general hope is that equations like

- [point]
- [projective line] = [line] + [point]
- [projective plane] = [plane] + [line] + [point]

can be put on increasingly solid mathematical footing with a deep meaning. Of course, the above equations are already known to be true in many senses, such as in the sense of CW-complex where "+" corresponds to attaching cells, and in the sense of various cohomology theories, where "+" corresponds to the direct sum.

From another viewpoint, motives continue the sequence of generalizations from rational functions on varieties to divisors on varieties to Chow groups of varieties. The generalization happens in more than one direction, since motives can be considered with respect to more types of equivalence than rational equivalence. The admissiable equivalences are given by the definition of an adequate equivalence relation.

Definition of pure motives

The category of pure motives often proceeds in three steps. Below we describe the case of Chow motives *Chow(k)*, where *k* is any field.

First step: category of (degree 0) correspondences, *Corr(k)*

The objects of *Corr(k)* are simply smooth projective varieties over *k*. The morphisms are correspondences. They generalize morphisms of varieties $X \to Y$, which can be associated with their graphs in $X \times Y$, to fixed dimensional Chow cycles on $X \times Y$.

It will be useful to describe correspondences of arbitrary degree, although morphisms in *Corr(k)* are correspondences of degree 0. In detail, let *X* and *Y* be smooth projective varieties, let $X = \coprod_i X_i$ be the decomposition of *X* into connected components, and let $d_i := \dim X_i$. If $r \in \mathbf{Z}$, then the correspondences of degree *r* from *X* to *Y* are

$$Corr^r(k)(X,Y) := \bigoplus_i A^{d_i+r}(X_i \times Y).$$

Correspondences are often denoted using the "⊢"-notation, e.g., $\alpha : X \vdash Y$. For any $\alpha \in Corr^r(X, Y)$ and $\beta \in Corr^s(Y, Z)$, their composition is defined by

$$\alpha \circ \beta := \pi_{XZ*}(\pi_{XY}^*(\alpha) \cdot \pi_{YZ}^*(\beta)) \in Corr^{r+s}(X,Z),$$

where the dot denotes the product in the Chow ring (i.e., intersection).

Returning to constructing the category *Corr(k)*, notice that the composition of degree 0 correspondences is degree 0. Hence we define morphisms of *Corr(k)* to be degree 0 correspondences.

The association,

$$F : \begin{array}{ccc} SmProj(k) & \longrightarrow & Corr(k) \\ X & \longmapsto & X \\ f & \longmapsto & \Gamma_f \end{array},$$

where $\Gamma_f \subset X \times Y$ is the graph of $f : X \to Y$, is a functor.

Just like *SmProj(k)*, the category *Corr(k)* has direct sums ($X \oplus Y := X \coprod Y$) and tensor products ($X \otimes Y := X \times Y$). It is a preadditive category (see the convention for preadditive vs. additive in the preadditive category article.) The sum of morphisms is defined by

$$\alpha + \beta := (\alpha, \beta) \in A^*(X \times X) \oplus A^*(Y \times Y) \hookrightarrow A^*((X \coprod Y) \times (X \coprod Y)).$$

Second step: category of pure effective Chow motives, *Chow^eff(k)*

The transition to motives is made by taking the pseudo-abelian envelope of *Corr(k)*:

$$Chow^{eff}(k) := Split(Corr(k)).$$

In other words, effective Chow motives are pairs of smooth projective varieties *X* and *idempotent* correspondences $\alpha : X \vdash X$, and morphisms are of a certain type of correspondence:

$$Ob(Chow^{eff}(k)) := \{(X,\alpha) \mid (\alpha : X \vdash X) \in Corr(k) \text{ such that } \alpha \circ \alpha = \alpha\}.$$

$$Mor((X,\alpha),(Y,\beta)) := \{f : X \vdash Y \mid f \circ \alpha = f = \beta \circ f\}.$$

Composition is the above defined composition of correspondences, and the identity morphism of (X, α) is defined to be $\alpha : X \vdash X$.

The association,

$$h : \begin{array}{ccc} SmProj(k) & \longrightarrow & Corr(k) \\ X & \longmapsto & [X] := (X, \Delta)_X \\ f & \longmapsto & [f] := \Gamma_f \subset X \times Y \end{array},$$

where $\Delta_X := [id_X]$ denotes the diagonal of $X \times X$, is a functor. The motive $[X]$ is often called the *motive associated to the variety* X.

As intended, $Chow^{eff}(k)$ is a pseudo-abelian category. The direct sum of effective motives is given by

$$([X], \alpha) \oplus ([Y], \beta) := ([X \coprod Y], \alpha + \beta),$$

The tensor product of effective motives is defined by

$$([X], \alpha) \otimes ([Y], \beta) := (X \times Y, \pi_X^* \alpha \cdot \pi_Y^* \beta), \qquad \pi_X : (X \times Y) \times (X \times Y) \to X \times X, \text{ and } \pi_Y : (X \times Y) \times (X \times$$

The tensor product of morphisms may also be defined. Let $f_1 : (X_1, \alpha_1) \to (Y_1, \beta_1)$ and $f_2 : (X_2, \alpha_2) \to (Y_2, \beta_2)$ be morphisms of motives. Then let $\gamma_1 \in A^*(X_1 \times Y_1)$ and $\gamma_1 \in A^*(X_1 \times Y_1)$ be representatives of f_1 and f_2. Then

$$f_1 \otimes f_2 : (X_1, \alpha_1) \otimes (X_2, \alpha_2) \vdash (Y_1, \beta_1) \otimes (Y_2, \beta_2), \qquad f_1 \otimes f_2 := \pi_1^* \gamma_1 \cdot \pi_2^* \gamma_2,$$

where $\pi_i : X_1 \times X_2 \times Y_1 \times Y_2 \to X_i \times Y_i$ are the projections.

Third step: category of pure Chow motives, *Chow(k)*

To proceed to motives, we adjoin to $Chow^{eff}(k)$ a formal inverse (with respect to the tensor product) of a motive called the Lefschetz motive. The effect is that motives become triples instead of pairs. The Lefschetz motive L is

$$L := (\mathbf{P}^1, \lambda), \qquad \lambda := pt \times \mathbf{P}^1 \in A^1(\mathbf{P}^1 \times \mathbf{P}^1).$$

If we define the motive **1**, called the *trivial Tate motive*, by **1** := h(Spec(k)), then the pleasant equation

$$[\mathbf{P}^1] = \mathbf{1} \oplus L$$

holds, since $\mathbf{1} \cong (\mathbf{P}^1, \mathbf{P}^1 \times pt)$. The tensor inverse of the Lefschetz motive is known as the *Tate motive*, $T := L^{-1}$. Then we define the category of pure Chow motives by

$$Chow(k) := Chow^{eff}(k)[T].$$

A motive is then a triple ($X \in SmProj(k)$, $p : X \vdash X$, $n \in \mathbf{Z}$) such that $p \,\hat{}\, p = p$. Morphisms are given by correspondences

$$f : (X, p, m) \to (Y, q, n), \quad f \in Corr^{n-m}(X, Y) \text{ such that } f \circ p = f = q \circ f,$$

and the composition of morphisms comes from composition of correspondences.

As intended, *Chow(k)* is a rigid pseudo-abelian category.

Other types of motives

In order to define an intersection product, cycles must be "movable" so we can intersect them in general position. Choosing an suitable equivalence relation on cycles will guarantee that every pair of cycles has an equivalent pair in general position that we can intersect. The Chow groups are defined using rational equivalence, but other equivalences are possible, and each defines a different sort of motive. Examples of equivalences, from strongest to weakest, are

- Rational equivalence
- Algebraic equivalence
- Smash-nilpotence equivalence (sometimes called Voevodsky equivalence)
- Homological equivalence (in the sense of Weil cohomology)
- Numerical equivalence

The literature occasionally calls every type of pure motive a Chow motive, in which case a motive with respect to algebraic equivalence would be called a *Chow motive modulo algebraic equivalence*.

Mixed motives

For a fixed base field k, the category of **mixed motives** is a conjectural abelian tensor category $MM(k)$, together with a contravariant functor

$$Var(k) \to MM(X)$$

taking values on all varieties (not just smooth projective ones as it was the case with pure motives). This should be such that motivic cohomology defined by

$$Ext^{*}_{\mathrm{MM}}(1, ?)$$

coincides with the one predicted by algebraic K-theory, and contains the category of Chow motives in a suitable sense (and other properties). The existence of such a category was conjectured by Beilinson. This category is yet to be constructed.

Instead of constructing such a category, it was proposed by Deligne to first construct a category *DM* having the properties one expects for the derived category

$$D^b(\mathrm{MM}(k)).$$

Getting *MM* back from *DM* would then be accomplished by a (conjectural) *motivic t-structure*.

The current state of the theory is that we do have a suitable category *DM*. Already this category is useful in applications. Voevodsky's Fields medal-winning proof of the Milnor conjecture uses these motives as a key ingredient.

There are different definitions due to Hanamura, Levine and Voevodsky. They are known to be equivalent in most cases and we will give Voevodsky's definition below. The category contains Chow motives as a full subcategory and gives the "right" motivic cohomology. However, Voevodsky also shows that (with integral coefficients) it does not admit a motivic t-structure.

- Start with the category *Sm* of smooth varieties over a perfect field. Similarly to the construction of pure motives above, instead of usual morphisms *smooth correspondences* are allowed. Compared to the (quite general) cycles used above, the definition of these correspondences is more restrictive; in particular they always intersect properly, so no moving of cycles and hence no equivalence relation is needed to get a well-defined composition of correspondences. This category is denoted *SmCor*, it is additive.
- As a technical intermediate step, take the bounded homotopy category $K^b(SmCor)$ of complexes of smooth schemes and correspondences.
- Apply localization of categories to force any variety X to be isomorphic to $X \times A^1$ (product with the affine line) and also, that a Mayer-Vietoris-sequence holds, i.e. $X = U \cup V$ (union of two open subvarieties) shall be isomorphic to $U \cap V \to U \sqcup V$.
- Finally, as above, take the pseudo-abelian envelope.

The resulting category is called the *category of effective geometric motives*. Again, formally inverting the Tate object, one gets the category *DM* of **geometric motives**.

Explanation for non-specialists

A commonly applied technique in mathematics is to study objects carrying a particular structure by introducing a category whose morphisms preserve this structure. Then one may ask, when are two given objects isomorphic and ask for a "particularly nice" representative in each isomorphism class. The classification of algebraic varieties, i.e. application of this idea in the case of algebraic varieties, is very difficult due to the highly non-linear structure of the objects. The relaxed question of studying varieties up to birational isomorphism has led to the field of birational geometry. Another way to handle the question is to attach to a given variety X an object of more linear nature, i.e. an object amenable to the techniques of linear algebra, for example a vector space. This "linearization" goes usually under the name of *cohomology*.

There are several important cohomology theories which reflect different structural aspects of varieties. The (partly conjectural) **theory of motives** is an attempt to find a universal way to linearize algebraic varieties, i.e. motives are supposed to provide a cohomology theory which embodies all these particular cohomologies. For example, the genus of a smooth projective curve C which is an interesting invariant of the curve, is an integer, which can be read off the dimension of the first Betti cohomology group of C. So, the motive of the curve should contain the genus information. Of course, the genus is a rather coarse invariant, so the motive of C is more than just this number.

The search for a universal cohomology

Each algebraic variety X has a corresponding motive $[X]$, so the simplest examples of motives are:

- [point]
- [projective line] = [point] + [line]
- [projective plane] = [plane] + [line] + [point]

These 'equations' hold in many situations, namely for de Rham cohomology and Betti cohomology, l-adic cohomology, the number of points over any finite field, and in multiplicative notation for local zeta-functions.

The general idea is that one **motive** has the same structure in any reasonable cohomology theory with good formal properties; in particular, any **Weil cohomology** theory will have such properties. There are different Weil cohomology theories, they apply in different situations and have values in different categories, and reflect different structural aspects of the variety in question:

- Betti cohomology is defined for varieties over (subfields of) the complex numbers, it has the advantage of being defined over the integers and is a topological invariant
- de Rham cohomology (for varieties over $\mathbb{C}$) comes with a mixed Hodge structure, it is a differential-geometric invariant
- l-adic cohomology (over any field of characteristic ≠ l) has a canonical Galois group action, i.e. has values in representations of the (absolute) Galois group
- crystalline cohomology

All these cohomology theories share common properties, e.g. existence of Mayer-Vietoris-sequences, homotopy invariance ($H^*(X) \cong H^*(X \times A^1)$, the product of X with the affine line) and others. Moreover, they are linked by comparison isomorphisms, for example Betti cohomology $H_{\text{Betti}}{}^*(X, \mathbb{Z}/n)$ of a smooth variety X over $\mathbb{C}$ with finite coefficients is isomorphic to l-adic cohomology with finite coefficients.

The **theory of motives** is an attempt to find a universal theory which embodies all these particular cohomologies and their structures and provides a framework for "equations" like

[projective line] = [line]+[point].

In particular, calculating the motive of any variety X directly gives all the information about the several Weil cohomology theories $H_{\text{Betti}}{}^*(X)$, $H_{\text{DR}}{}^*(X)$ etc.

Beginning with Grothendieck, people have tried to precisely define this theory for many years.

Motivic cohomology

Motivic cohomology itself had been invented before the creation of mixed motives by means of algebraic K-theory. The above category provides a neat way to (re)define it by

$$H^n(X, m) := H^n(X, \mathbb{Z}(m)) := \mathrm{Hom}_{DM}(X, \mathbb{Z}(m)[n]),$$

where n and m are integers and $\mathbb{Z}(m)$ is the m-th tensor power of the Tate object $\mathbb{Z}(1)$, which in Voevodsky's setting is the complex $\mathbb{P}^1 \to$ *point* shifted by *-2*, and $[n]$ means the usual shift in the triangulated category.

Conjectures related to motives

The standard conjectures were first formulated in terms of the interplay of algebraic cycles and Weil cohomology theories. The category of pure motives provides a categorical framework for these conjectures.

The standard conjectures are commonly considered to be very hard and are open in the general case. Grothendieck, with Bombieri, showed the depth of the motivic approach by producing a conditional (very short and elegant) proof of the Weil conjectures (which are proven by different means by Deligne), assuming the standard conjectures to hold.

For example, the *Künneth standard conjecture*, which states the existence of algebraic cycles $\pi^i \subset X \times X$ inducing the canonical projectors $H^*(X) \twoheadrightarrow H^i(X) \hookrightarrow H^*(X)$ (for any Weil cohomology H) implies that every pure motive M decomposes in graded pieces of weight n: $M = \oplus Gr_n M$. The terminology *weights* comes from a similar decomposition of, say, de-Rham cohomology of smooth projective varieties, see Hodge theory.

Conjecture D, stating the concordance of numerical and homological equivalence, implies the equivalence of pure motives with respect to homological and numerical equivalence. (In particular the former category of motives would not depend on the choice of the Weil cohomology theory). Jannsen (1992) proved the following unconditional result: the category of (pure) motives over a field is abelian and semisimple if and only if the chosen equivalence relation is numerical equivalence.

The Hodge conjecture, may be neatly reformulated using motives: it holds iff the *Hodge realization* mapping any pure motive with rational coefficients (over a subfield k of $\mathbb{C}$) to its Hodge structure is a full functor $H : \mathrm{M}(k)_{\mathbb{Q}} \to HS_{\mathbb{Q}}$ (rational Hodge structures). Here pure motive means pure motive with respect to homological equivalence.

Similarly, the Tate conjecture is equivalent to: the so-called Tate realization, i.e. ℓ-adic cohomology is a faithful functor $H : M(k)_{\mathbb{Q}_\ell} \to Rep_{\ell}(Gal(k))$ (pure motives up to homological equivalence, continuous representations of the absolute Galois group of the base field k), which takes values in semi-simple representations. (The latter part is automatic in the case of the Hodge analogue).

Tannakian formalism and motivic Galois group

To motivate the (conjectural) motivic Galois group, fix a field k and consider the functor

finite separable extensions K of k $\to$ *finite sets with a (continuous) action of the absolute Galois group of k*

which maps K to the (finite) set of embeddings of K into an algebraic closure of k. In Galois theory this functor is shown to be an equivalence of categories. Notice that fields are *0*-dimensional. Motives of this kind are called *Artin motives*. By $\mathbb{Q}$-linearizing the above objects, another way of expressing the above is to say that Artin motives are equivalent to finite $\mathbb{Q}$-vector spaces together with an action of the Galois group.

The objective of the **motivic Galois group** is to extend the above equivalence to higher-dimensional varieties. In order to do this, the technical machinery of Tannakian category theory (going back to Tannaka-Krein duality, but a purely algebraic theory) is used. Its purpose is to shed light on both the Hodge conjecture and the Tate conjecture, the outstanding questions in algebraic cycle theory. Fix a Weil cohomology theory H. It gives a functor from M_{num} (pure motives using numerical equivalence) to finite-dimensional $\mathbb{Q}$-vector spaces. It can be shown that the former category is a Tannakian category. Assuming the equivalence of homological and numerical equivalence, i.e. the above standard conjecture *D*, the functor *H* is an exact faithful tensor-functor. Applying the Tannakian formalism, one concludes that M_{num} is equivalent to the category of representations of an algebraic group G, which is called motivic Galois group.

It is to the theory of motives what the Mumford-Tate group is to Hodge theory. Again speaking in rough terms, the Hodge and Tate conjectures are types of invariant theory (the spaces that are morally the algebraic cycles are picked out by invariance under a group, if one sets up the correct definitions). The motivic Galois group has the surrounding representation theory. (What it is not, is a Galois group; however in terms of the Tate conjecture and Galois representations on étale cohomology, it predicts the image of the Galois group, or, more accurately, its Lie algebra.)

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Grothendieck–Hirzebruch–Riemann–Roch theorem

In mathematics, specifically in algebraic geometry, the **Grothendieck–Riemann–Roch theorem** is a far-reaching result on coherent cohomology. It is a generalisation of the Hirzebruch–Riemann–Roch theorem, about complex manifolds, which is itself a generalisation of the classical Riemann–Roch theorem for line bundles on compact Riemann surfaces.

Riemann–Roch type theorems relate Euler characteristics of the cohomology of a vector bundle with their topological degrees, or more generally their characteristic classes in (co)homology or algebraic analogues thereof. The classical Riemann–Roch theorem does this for curves and line bundles, whereas the Hirzebruch–Riemann–Roch theorem generalises this to vector bundles over manifolds. The Grothendieck–Hirzebruch–Riemann–Roch theorem sets both theorems in a relative situation of a morphism between two manifolds (or more general schemes) and changes the theorem from a statement about a single bundle, to one applying to chain complexes of sheaves.

The theorem has been very influential, not least for the development of the Atiyah–Singer index theorem. Conversely, complex analytic analogues of the Grothendieck–Hirzebruch–Riemann–Roch theorem can be proved using the families index theorem. Alexander Grothendieck, its author, was rumored to have finished the proof around 1956 but did not publish his theorem because he was not satisfied with it. Instead Armand Borel and Jean-Pierre Serre wrote up and published Grothendieck's preliminary (as he saw it) proof.

Formulation

Let *X* be a smooth quasi-projective scheme over a field. Under these assumptions, the Grothendieck group

$$K_0(X)$$

of bounded complexes of coherent sheaves is canonically isomorphic to the Grothendieck group of bounded complexes of finite-rank vector bundles. Using this isomorphism, consider the Chern character (a rational combination of Chern classes) as a functorial transformation

$$\mathrm{ch}\colon K_0(X) \to A(X, \mathbb{Q})$$

where

$$A_d(X, \mathbb{Q})$$

is the Chow group of cycles on *X* of dimension *d* modulo rational equivalence, tensored with the rational numbers. In case *X* is defined over the complex numbers, the latter group maps to the topological cohomology group

$$H^{2\dim(X)-2d}(X, \mathbb{Q}).$$

Now consider a proper morphism

$$f\colon X \to Y$$

between smooth quasi-projective schemes and a bounded complex of sheaves $\mathcal{F}^\bullet$.

The **Grothendieck–Riemann–Roch theorem** relates the push forward maps

$$f_! = \sum (-1)^i R^i f_*\colon K_0(X) \to K_0(Y)$$

and the pushforward

$$f_*: A(X) \to A(Y),$$

by the formula

$$\mathrm{ch}(f_! \mathcal{F}^\bullet)\mathrm{td}(Y) = f_*(\mathrm{ch}(\mathcal{F}^\bullet)\mathrm{td}(X)).$$

Here td(X) is the Todd genus of (the tangent bundle of) X. Thus the theorem gives a precise measure for the lack of commutativity of taking the push forwards in the above senses and the Chern character and shows that the needed correction factors depends on X and Y only. In fact, since the Todd genus is functorial and multiplicative in exact sequences, we can rewrite the Grothendieck Hirzebruch Riemann Roch formula to

$$\mathrm{ch}(f_! \mathcal{F}^\bullet) = f_*(\mathrm{ch}(\mathcal{F}^\bullet)\mathrm{td}(T_f))$$

where T_f is the relative tangent sheaf of f. This is often useful in applications, for example if f is a locally trivial fibration.

Generalising and specialising

Generalisations of the theorem can be made to the non-smooth case by considering an appropriate generalisation of the combination ch(—)td(X) and to the non-proper case by considering cohomology with compact support.

The arithmetic Riemann–Roch theorem extends the Grothendieck–Riemann–Roch theorem to arithmetic schemes.

The Hirzebruch–Riemann–Roch theorem is (essentially) the special case where Y is a point and the field is the field of complex numbers.

History

Grothendieck's version of the Riemann–Roch theorem was originally conveyed in a letter to Serre around 1956–7. It was made public at the initial Bonn Arbeitstagung, in 1957. Serre and Armand Borel subsequently organized a seminar at Princeton to understand it. The final published paper was in effect the Borel–Serre exposition.

The significance of Grothendieck's approach rests on several points. First, Grothendieck changed the statement itself: the theorem was, at the time, understood to be a theorem about a variety, whereas after Grothendieck, it was known to essentially be understood as a theorem about a morphism between varieties. In short, he applied a strong categorical approach to a hard piece of analysis. Moreover, Grothendieck introduced K-groups, as discussed above, which paved the way for algebraic K theory.

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Coherent sheaf

In mathematics, especially in algebraic geometry and the theory of complex manifolds, **coherent sheaves** are a specific class of sheaves having particularly manageable properties closely linked to the geometrical properties of the underlying space. The definition of coherent sheaves is made with reference to a sheaf of rings that codifies this geometrical information. In addition, there is a related concept of **quasi-coherent sheaves**. Many results and properties in algebraic geometry and complex analytic geometry are formulated in terms of coherent sheaves and their cohomology.

Coherent sheaves can be seen as a generalization of (sheaves of sections of) vector bundles. They form a category closed under usual operations such as taking kernels, cokernels and finite direct sums. In addition, under suitable compactness conditions they are preserved under maps of the underlying spaces and have finite dimensional cohomology spaces.

Definition

A *coherent sheaf* on a ringed space $(X, \mathcal{O}_X)$ is a sheaf $\mathcal{F}$ of $\mathcal{O}_X$ -modules with the following two properties:

1. $\mathcal{F}$ is of *finite type* over $\mathcal{O}_X$, i.e., for any point $x \in X$ there is an open neighbourhood $U \subset X$ such that the restriction $\mathcal{F}|_U$ of $\mathcal{F}$ to U is generated by a finite number of sections (in other words, there is a surjective morphism $\mathcal{O}_X^n|_U \to \mathcal{F}|_U$ for some $n \in \mathbb{N}$); and
2. for any open set $U \subset X$, any $n \in \mathbb{N}$ and any morphism $\phi\colon \mathcal{O}_X^n|_U \to \mathcal{F}|_U$ of $\mathcal{O}_X$ -modules, the kernel of ϕ is of finite type.

The sheaf of rings $\mathcal{O}_X$ is coherent if it is coherent considered as a sheaf of modules over itself. Important examples of coherent sheaves of rings include the sheaf of germs of holomorphic functions on a complex manifolds and the structure sheaf of a Noetherian scheme from algebraic geometry.

A coherent sheaf is always a sheaf of *finite presentation*, or in other words each point $x \in X$ has an open neighbourhood U such that the restriction $\mathcal{F}|_U$ of $\mathcal{F}$ to U is isomorphic to the cokernel of a morphism $\mathcal{O}_X^n|_U \to \mathcal{O}_X^m|_U$ for some integers n and m . If $\mathcal{O}_X$ is coherent, then the converse is true and each sheaf of finite presentation over $\mathcal{O}_X$ is coherent.

For a sheaf of rings $\mathcal{O}$, a sheaf $\mathcal{F}$ of $\mathcal{O}$ -modules is said to be **quasi-coherent** if it has a local presentation, i.e. if there exist an open cover by U_i of the topological space and an exact sequence

$$\mathcal{O}^{(I_i)}|_{U_i} \to \mathcal{O}^{(J_i)}|_{U_i} \to \mathcal{F}|_{U_i} \to 0$$

where the first two terms of the sequence are direct sums (possibly infinite) of copies of the structure sheaf.

For an affine variety X with (affine) coordinate ring R, there exists a covariant equivalence of categories between that of quasi-coherent sheaves and sheaf morphisms on the one hand, and R-modules and module homomorphisms on the other hand. In case the ring R is Noetherian, coherent sheaves correspond exactly to finitely generated modules.

Coherence of sheaves is working in the background of some results in commutative algebra, e.g. Nakayama's lemma, which in terms of sheaves says that if $\mathcal{F}$ is a coherent sheaf, then the fiber $k(x) \otimes_{\mathcal{O}_{X,x}} \mathcal{F}_x = 0$ if and only if there is a neighborhood U of x so that $\mathcal{F}|_U = 0$.

The role played by coherent sheaves is as a class of sheaves, say on an algebraic variety or complex manifold, that is more general than the locally free sheaf — such as invertible sheaf, or sheaf of sections of a (holomorphic) vector bundle — but still with manageable properties. The generality is desirable, to be able to take kernels and cokernels of morphisms, for example, without moving outside the given class of sheaves.

Examples of coherent sheaves

- On noetherian schemes, the structure sheaf O_X itself.
- Sheaves of sections in vector bundles.
- The Oka coherence theorem shows that the sheaf of holomorphic functions on a complex manifold is coherent.
- Ideal sheaves: If Z is a closed complex subspace of a complex analytic space X, the sheaf I_Z of all holomorphic functions vanishing on Z is coherent.
- Structure sheaves of subspaces.

Coherent cohomology

The sheaf cohomology theory of coherent sheaves is called ***coherent cohomology***. It is one of the major and most fruitful applications of sheaves, and its results connect quickly with classical theories.

Using a theorem of Schwartz on compact operators in Frechet spaces, Cartan and Serre proved that compact complex manifolds have the property that their sheaf cohomology for any coherent sheaf consists of vector spaces of finite dimension. This result had been proved previously by Kodaira for the particular case of locally free sheaves on Kähler manifolds. It plays a major role in the proof of the "GAGA" equivalence analytic <-> algebraic. An algebraic (and much easier) version of this theorem was proved by Serre. Relative versions of this result for a proper morphism were proved by Grothendieck in the algebraic case and by Grauert and Remmert in the analytic case. For example Grothendieck's result concerns the functor Rf_* or push-forward, in sheaf cohomology. (It is the right derived functor of the direct image of a sheaf.) For a proper morphism in the sense of scheme theory, it was shown that this functor sends coherent sheaves to coherent sheaves. The result of Serre is the case of a morphism to a point.

The duality theory in scheme theory that extends Serre duality is called coherent duality (or *Grothendieck duality*). Under some mild conditions of finiteness, the sheaf of Kähler differentials on an algebraic variety is a coherent sheaf Ω^1. When the variety is non-singular its 'top' exterior power acts as the *dualising object*; and it is locally free (effectively it is the sheaf of sections of the cotangent bundle, when working over the complex numbers, but that is a statement that requires more precision since only *holomorphic* 1-forms count as sections). The successful extension of the theory beyond this case was a major step.

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Grothendieck topology

In category theory, a branch of mathematics, a **Grothendieck topology** is a structure on a category *C* which makes the objects of *C* act like the open sets of a topological space. A category together with a choice of Grothendieck topology is called a **site**.

Grothendieck topologies axiomatize the notion of an open cover. Using the notion of covering provided by a Grothendieck topology, it becomes possible to define sheaves on a category and their cohomology. This was first done in algebraic geometry and algebraic number theory by Alexander Grothendieck to define the étale cohomology of a scheme. It has been used to define other cohomology theories since then, such as l-adic cohomology, flat cohomology, and crystalline cohomology. While Grothendieck topologies are most often used to define cohomology theories, they have found other applications as well, such as to John Tate's theory of rigid analytic geometry.

There is a natural way to associate a site to an ordinary topological space, and Grothendieck's theory is loosely regarded as a generalization of classical topology. Under meager point-set hypotheses, namely sobriety, this is completely accurate—it is possible to recover a sober space from its associated site. However simple examples such as the indiscrete topological space show that not all topological spaces can be expressed using Grothendieck topologies. Conversely, there are Grothendieck topologies which do not come from topological spaces.

Introduction

André Weil's famous Weil conjectures proposed that certain properties of equations with integral coefficients should be understood as geometric properties of the algebraic variety that they defined. His conjectures postulated that there should be a cohomology theory of algebraic varieties which gave number-theoretic information about their defining equations. This cohomology theory was known as the "Weil cohomology", but using the tools he had available, Weil was unable to construct it.

In the early 1960s, Alexander Grothendieck introduced étale maps into algebraic geometry as algebraic analogues of local analytic isomorphisms in analytic geometry. He used étale coverings to define an algebraic analogue of the fundamental group of a topological space. Soon Jean-Pierre Serre noticed that some properties of étale coverings mimicked those of open immersions, and that consequently it was possible to make constructions which imitated the cohomology functor H^1. Grothendieck saw that it would be possible to use Serre's idea to define a cohomology theory which he suspected would be the Weil cohomology. To define this cohomology theory, Grothendieck needed to replace the usual, topological notion of an open covering with one that would use étale coverings instead. Grothendieck also saw how to phrase the definition of covering abstractly; this is where the definition of a Grothendieck topology comes from.

Definition

Motivation

The classical definition of a sheaf begins with a topological space X. A sheaf associates information to the open sets of X. This information can be phrased abstractly by letting $O(X)$ be the category whose objects are the open subsets U of X and whose morphisms are the inclusion maps $V \to U$ of open sets U and V of X. We will call such maps *open immersions*, just as in the context of schemes. Then a presheaf on X is a contravariant functor from $O(X)$ to the category of sets, and a sheaf is a presheaf which satisfies the gluing axiom. The gluing axiom is phrased in terms of pointwise covering, i.e., $\{U_i\}$ covers U if and only if $\cup_i U_i = U$. In this definition, U_i is an open subset of X. Grothendieck topologies replace each U_i with an entire family of open subsets; in this example, U_i is replaced by the family of all open immersions $V_{ij} \to U_i$. Such a collection is called a *sieve*. Pointwise covering is replaced by the notion of a *covering family*; in the above example, the set of all $\{V_{ij} \to U_i\}_j$ as i varies is a covering family of U. Sieves and covering families can be axiomatized, and once this is done open sets and pointwise covering can be replaced by other notions which describe other properties of the space X.

Sieves

In a Grothendieck topology, the notion of a collection of open subsets of U stable under inclusion is replaced by the notion of a sieve. If c is any given object in C, a **sieve** on c is a subfunctor of the functor Hom(−, c); (this is the Yoneda embedding applied to c). In the case of $O(X)$, a sieve S on an open set U selects a collection of open subsets of U which is stable under inclusion. More precisely, consider that for any open subset V of U, $S(V)$ will be a subset of Hom(V, U), which has only one element, the open immersion $V \to U$. Then V will be considered "selected" by S if and only if $S(V)$ is nonempty. If W is a subset of V, then there is a morphism $S(V) \to S(W)$ given by composition with the inclusion $W \to V$. If $S(V)$ is non-empty, it follows that $S(W)$ is also non-empty.

If S is a sieve on X, and f: $Y \to X$ is a morphism, then left composition by f gives a sieve on Y called the **pullback of** S **along** f, denoted by $f{*}S$. It is defined as the fibered product $S \times_{\mathrm{Hom}(-,\, X)} \mathrm{Hom}(-, Y)$ together with its natural embedding in Hom(−, Y). More concretely, for each object Z of C, $f{*}S(Z) = \{\ g\colon Z \to Y \mid fg \in S(Z)\ \}$, and $f{*}S$ inherits its action on morphisms by being a subfunctor of Hom(−, Y). In the classical example, the pullback of a collection $\{V_i\}$ of subsets of U along an inclusion $W \to U$ is the collection $\{V_i \cap \mathrm{W}\}$.

Covering Sieves

A classical topology on a set X is a collection of distinguished subsets, called open sets. This selection is subject to certain conditions: the axioms of a topological space. By comparison, a Grothendieck topology J on a category C is a collection, *for each object c of C*, of distinguished sieves on c, called **covering sieves** of c and denoted by $J(c)$. This selection will be subject to certain axioms, stated below. Continuing the previous example, a sieve S on an open set U in $O(X)$ will be a covering sieve if and only if the union of all the open sets V for which $S(V)$ is nonempty equals U; in other words, if and only if S gives us a collection of open sets which cover U in the classical sense.

Axioms

The conditions we impose on a **Grothendieck topology** are:

- (T 1) (Base change) If S is a covering sieve on X, and f: $Y \to X$ is a morphism, then the pullback $f{*}S$ is a covering sieve on Y.
- (T 2) (Local character) Let S be a covering sieve on X, and let T be any sieve on X. Suppose that for each object Y of C and each arrow f: $Y \to X$ in $S(Y)$, the pullback sieve $f{*}T$ is a covering sieve on Y. Then T is a covering sieve on X.
- (T 3) (Identity) Hom(−, X) is a covering sieve on X for any object X in C.

The base change axiom corresponds to the idea that if $\{ U_i \}$ covers U, then $\{U_i \cap V\}$ should cover $U \cap V$. The local character axiom corresponds to the idea that if $\{U_i\}$ covers U and $\{V_{ij}\}_{j \in J_i}$ covers U_i for each i, then the collection $\{V_{ij}\}$ for all i and j should cover U. Lastly, the identity axiom corresponds to the idea that any set is covered by all its possible subsets.

Alternative Axioms

In fact, it is possible to put these axioms in another form where their geometric character is more apparent, assuming that the underlying category C contains certain fibered products. In this case, instead of specifying sieves, we can specify that certain collections of maps with a common codomain should cover their codomain. These collections are called **covering families**. If the collection of all covering families satisfies certain axioms, then we say that they form a **Grothendieck pretopology**. These axioms are:

- (PT 0) (Existence of fibered products) For all objects X of C, and for all morphisms $X_0 \to X$ which appear in some covering family of X, and for all morphisms $Y \to X$, the fibered product $X_0 \times_X Y$ exists.
- (PT 1) (Stability under base change) For all objects X of C, all morphisms $Y \to X$, and all covering families $\{X_\alpha \to X\}$, the family $\{X_\alpha \times_X Y \to Y\}$ is a covering family.
- (PT 2) (Local character) If $\{X_\alpha \to X\}$ is a covering family, and if for all α, $\{X_{\beta\alpha} \to X_\alpha\}$ is a covering family, then the family of composites $\{X_{\beta\alpha} \to X_\alpha \to X\}$ is a covering family.
- (PT 3) (Isomorphisms) If f: $Y \to X$ is an isomorphism, then $\{f\}$ is a covering family.

For any pretopology, the collection of all sieves that contain a covering family from the pretopology is always a Grothendieck topology.

For categories with fibered products, there is a converse. Given a collection of arrows $\{X_\alpha \to X\}$, we construct a sieve S by letting $S(Y)$ be the set of all morphisms $Y \to X$ that factor through some arrow $X_\alpha \to X$. This is called the sieve **generated by** $\{X_\alpha \to X\}$. Now choose a topology. Say that $\{X_\alpha \to X\}$ is a covering family if and only if the sieve that it generates is a covering sieve for the given topology. It is easy to check that this defines a pretopology.

(PT 3) is sometimes replaced by a weaker axiom:

- (PT 3') (Identity) If $1_X : X \to X$ is the identity arrow, then $\{1_X\}$ is a covering family.

(PT 3) implies (PT 3'), but not conversely. However, suppose that we have a collection of covering families that satisfies (PT 0) through (PT 2) and (PT 3'), but not (PT 3). These families generate a pretopology. The topology generated by the original collection of covering families is then the same as the topology generated by the pretopology, because the sieve generated by an isomorphism $Y \to X$ is Hom(−, X). Consequently, if we restrict our attention to topologies, (PT 3) and (PT 3') are equivalent.

Sites and sheaves

Let C be a category and let J be a Grothendieck topology on C. The pair (C, J) is called a **site**.

A **presheaf** on a category is a contravariant functor from C to the category of all sets. Note that for this definition C is not required to have a topology. A sheaf on a site, however, should allow gluing, just like sheaves in classical topology. Consequently, we define a **sheaf** on a site to be a presheaf F such that for all objects X and all covering sieves S on X, the natural map Hom(Hom(−, X), F) → Hom(S, F) induced by the inclusion of S into Hom(−, X) is a bijection. Halfway in between a presheaf and a sheaf is the notion of a **separated presheaf**, where the natural map above is required to be only an injection, not a bijection, for all sieves S. A **morphism** of presheaves or of sheaves is a natural transformation of functors. The category of all sheaves on C is the **topos** defined by the site (C, J).

Using the Yoneda lemma, it is possible to show that a presheaf on the category $O(X)$ is a sheaf on the topology defined above if and only if it is a sheaf in the classical sense.

Sheaves on a pretopology have a particularly simple description: For each covering family $\{X_\alpha \to X\}$, the diagram

$$F(X) \to \prod_{\alpha \in A} F(X_\alpha) \rightrightarrows \prod_{\alpha,\beta \in A} F(X_\alpha \times_X X_\beta)$$

must be an equalizer. For a separated presheaf, the first arrow need only be injective.

Similarly, one can define presheaves and sheaves of abelian groups, rings, modules, and so on. One can require either that a presheaf *F* is a contravariant functor to the category of abelian groups (or rings, or modules, etc.), or that *F* be an abelian group (ring, module, etc.) object in the category of all contravariant functors from *C* to the category of sets. These two definitions are equivalent.

Examples of sites

The discrete and indiscrete topologies

Let **C** be any category. To define the **discrete topology**, we declare all sieves to be covering sieves. If **C** has all fibered products, this is equivalent to declaring all families to be covering families. To define the **indiscrete topology**, we declare only the sieves of the form Hom(−, *X*) to be covering sieves. The indiscrete topology is also known as the **biggest** or **chaotic** topology, and it is generated by the pretopology which has only isomorphisms for covering families. A sheaf on the indiscrete site is the same thing as a presheaf.

The canonical topology

Let **C** be any category. The Yoneda embedding gives a functor Hom(−, *X*) for each object *X* of **C**. The **canonical topology** is the biggest topology such that every representable presheaf Hom(−, *X*) is a sheaf. A covering sieve or covering family for this site is said to be *strictly universally epimorphic*. A topology which is less fine than the canonical topology, that is, for which every covering sieve is strictly universally epimorphic, is called **subcanonical**. Subcanonical sites are exactly the sites for which every presheaf of the form Hom(−, *X*) is a sheaf. Most sites encountered in practice are subcanonical.

Small site associated to a topological space

We repeat the example which we began with above. Let *X* be a topological space. We defined *O*(*X*) to be the category whose objects are the open sets of *X* and whose morphisms are inclusions of open sets. The covering sieves on an object *U* of *O*(*X*) were those sieves *S* satisfying the following condition:

- If *W* is the union of all the sets *V* such that *S*(*V*) is non-empty, then *W* = *U*.

This topology can also naturally be expressed as a pretopology. We say that a family of inclusions $\{V_\alpha \subseteq U\}$ is a covering family if and only if the union $\cup V_\alpha$ equals *U*. This site is called the **small site associated to a topological space *X*.**

Big site associated to a topological space

Let *Spc* be the category of all topological spaces. Given any family of functions $\{u_\alpha : V_\alpha \to X\}$, we say that it is a **surjective family** or that the morphisms u_α are **jointly surjective** if $\cup\ u_\alpha(V_\alpha)$ equals *X*. We define a pretopology on *Spc* by taking the covering families to be surjective families all of whose members are open immersions. Let *S* be a sieve on *Spc*. *S* is a covering sieve for this topology if and only if:

- For all *Y* and every morphism $f : Y \to X$ in *S*(*Y*), there exists a *V* and a $g : V \to X$ such that *g* is an open immersion, *g* is in *S*(*V*), and *f* factors through *g*.
- If *W* is the union of all the sets *f*(*Y*), where $f : Y \to X$ is in *S*(*Y*), then *W* = *X*.

Fix a topological space *X*. Consider the comma category *Spc/X* of topological spaces with a fixed continuous map to *X*. The topology on *Spc* induces a topology on *Spc/X*. The covering sieves and covering families are almost exactly the same; the only difference is that now all the maps involved commute with the fixed maps to *X*. This is the **big**

site associated to a topological space X . Notice that *Spc* is the big site associated to the one point space. This site was first considered by Jean Giraud.

The big and small sites of a manifold

Let M be a manifold. M has a category of open sets $O(M)$ because it is a topological space, and it gets a topology as in the above example. For two open sets U and V of M, the fiber product $U \times_M V$ is the open set $U \cap V$, which is still in $O(M)$. This means that the topology on $O(M)$ is defined by a pretopology, the same pretopology as before.

Let *Mfd* be the category of all manifolds and continuous maps. (Or smooth manifolds and smooth maps, or real analytic manifolds and analytic maps, etc.) *Mfd* is a subcategory of *Spc*, and open immersions are continuous (or smooth, or analytic, etc.), so *Mfd* inherits a topology from *Spc*. This lets us construct the big site of the manifold M as the site *Mfd/M*. We can also define this topology using the same pretopology we used above. Notice that to satisfy (PT 0), we need to check that for any continuous map of manifolds $X \to Y$ and any open subset U of Y, the fibered product $U \times_Y X$ is in *Mfd/M*. This is just the statement that the preimage of an open set is open. Notice, however, that not all fibered products exist in *Mfd* because the preimage of a smooth map at a critical value need not be a manifold.

Topologies on the category of schemes

The category of schemes, denoted *Sch*, has a tremendous number of useful topologies. A complete understanding of some questions may require examining a scheme using several different topologies. All of these topologies have associated small and big sites. The big site is formed by taking the entire category of schemes and their morphisms, together with the covering sieves specified by the topology. The small site over a given scheme is formed by only taking the objects and morphisms which are part of a cover of the given scheme.

The most elementary of these is the Zariski topology. Let X be a scheme. X has an underlying topological space, and this topological space determines a Grothendieck topology. The Zariski topology on *Sch* is generated by the pretopology whose covering families are jointly surjective families of scheme-theoretic open immersions. The covering sieves S for *Zar* are characterized by the following two properties:

- For all Y and every morphism $f : Y \to X$ in $S(Y)$, there exists a V and a $g : V \to X$ such that g is an open immersion, g is in $S(V)$, and f factors through g.
- If W is the union of all the sets $f(Y)$, where $f : Y \to X$ is in $S(Y)$, then $W = X$.

Despite their outward similarities, the topology on *Zar* is *not* the restriction of the topology on *Spc*! This is because there are morphisms of schemes which are topologically open immersions but which are not scheme-theoretic open immersions. For example, let A be a non-reduced ring and let N be its ideal of nilpotents. The quotient map $A \to A/N$ induces a map Spec $A/N \to$ Spec A which is the identity on underlying topological spaces. To be a scheme-theoretic open immersion it must also induce an isomorphism on structure sheaves, which this map does not do. In fact, this map is a closed immersion.

The étale topology is finer than the Zariski topology. It was the first Grothendieck topology to be closely studied. Its covering families are jointly surjective families of étale morphisms. It is finer than the Nisnevich topology, but neither finer nor coarser than the *cdh* and l′ topologies.

There are two flat topologies, the *fppf* topology and the *fpqc* topology. In the *fppf* topology, covering morphisms are finitely presented and faithfully flat, and in the *fpqc* topology, covering morphisms are finitely presented and quasi-compact. These topologies are closely related to descent. The *fpqc* topology is finer than all the topologies mentioned above, and it is very close to the canonical topology.

Grothendieck introduced crystalline cohomology to study the p-torsion part of the cohomology of characteristic p varieties. In the *crystalline topology* which is the basis of this theory, covering maps are given by infinitesimal thickenings together with divided power structures. The crystalline covers of a fixed scheme form a category with no final object.

Continuous and cocontinuous functors

There are two natural types of functors between sites. They are given by functors which are compatible with the topology in a certain sense.

Continuous functors

If (C, J) and (D, K) are sites and $u : C \to D$ is a functor, then u is **continuous** if for every sheaf F on D with respect to the topology K, the presheaf Fu is a sheaf with respect to the topology J. Continuous functors induce functors between the corresponding topoi by sending a sheaf F to Fu. These functors are called **pushforwards**. If $\tilde{C}$ and $\tilde{D}$ denote the topoi associated to C and D, then the pushforward functor is $u_s : \tilde{D} \to \tilde{C}$.

u_s admits a left adjoint u^s called the **pullback**. u^s need not preserve limits, even finite limits.

In the same way, u sends a sieve on an object X of C to a sieve on the object uX of D. A continuous functor sends covering sieves to covering sieves. If J is the topology defined by a pretopology, and if u commutes with fibered products, then u is continuous if and only if it sends covering sieves to covering sieves and if and only if it sends covering families to covering families. In general, it is *not* sufficient for u to send covering sieves to covering sieves (see SGA IV 3, Exemple 1.9.3).

Cocontinuous functors

Again, let (C, J) and (D, K) be sites and $v : C \to D$ be a functor. If X is an object of C and R is a sieve on vX, then R can be pulled back to a sieve S as follows: A morphism $f : Z \to X$ is in S if and only if $v(f) : vZ \to vX$ is in R. This defines a sieve. v is **cocontinuous** if and only if for every object X of C and every covering sieve R of vX, the pullback S of R is a covering sieve on X.

Composition with v sends a presheaf F on D to a presheaf Fv on C, but if v is cocontinuous, this need not send sheaves to sheaves. However, this functor on presheaf categories, usually denoted $\hat{v}^*$, admits a right adjoint $\hat{v}_*$. Then v is cocontinuous if and only if $\hat{v}_*$ sends sheaves to sheaves, that is, if and only if it restricts to a functor $v_* : \tilde{C} \to \tilde{D}$. In this case, the composite of $\hat{v}^*$ with the associated sheaf functor is a left adjoint of v_* denoted v^*. Furthermore, v^* preserves finite limits, so the adjoint functors v_* and v^* determine a geometric morphism of topoi $\tilde{C} \to \tilde{D}$.

Morphisms of sites

A continuous functor $u : C \to D$ is a **morphism of sites** $D \to C$ (*not* $C \to D$) if u^s preserves finite limits. In this case, u^s and u_s determine a geometric morphism of topoi $\tilde{C} \to \tilde{D}$. The reasoning behind the convention that a continuous functor $C \to D$ is said to determine a morphism of sites in the opposite direction is that this agrees with the intuition coming from the case of topological spaces. A continuous map of topological spaces $X \to Y$ determines a continuous functor $O(Y) \to O(X)$. Since the original map on topological spaces is said to send X to Y, the morphism of sites is said to as well.

A particular case of this happens when a continuous functor admits a left adjoint. Suppose that $u : C \to D$ and $v : D \to C$ are functors with u right adjoint to v. Then u is continuous if and only if v is cocontinuous, and when this happens, u^s is naturally isomorphic to v^* and u_s is naturally isomorphic to v_*. In particular, u is a morphism of sites.

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Crystalline cohomology

In mathematics, **crystalline cohomology** is a Weil cohomology theory for schemes introduced by Alexander Grothendieck (1966, 1968) and developed by Pierre Berthelot (1974). Its values are modules over rings of Witt vectors over the base field.

Crystalline cohomology is partly inspired by the p-adic proof in Dwork (1960) of part of the Weil conjectures and is closely related to the (algebraic) **de Rham cohomology** introduced by Grothendieck (1963). Roughly speaking, crystalline cohomology of a variety X in characteristic p is the de Rham cohomology of a smooth lift of X to characteristic 0, while de Rham cohomology of X is the crystalline cohomology reduced mod p (after taking into account higher *Tor*s).

The idea of crystalline cohomology, roughly, is to replace the Zariski open sets of a scheme by infinitesimal thickenings of Zariski open sets with divided power structures. The motivation for this is that it can then be calculated by taking a local lifting of a scheme from characteristic p to characteristic 0 and employing an appropriate version of algebraic de Rham cohomology.

Crystalline cohomology only works well for smooth proper schemes. Rigid cohomology extends it to more general schemes.

Applications

For schemes in characteristic p, crystalline cohomology theory can handle questions about p-torsion in cohomology groups better than p-adic étale cohomology. This makes it a natural backdrop for much of the work on p-adic L-functions.

Crystalline cohomology, from the point of view of number theory, fills a gap in the l-adic cohomology information, which occurs exactly where there are 'equal characteristic primes'. Traditionally the preserve of ramification theory, crystalline cohomology converts this situation into Dieudonné module theory, giving an important handle on arithmetic problems. Conjectures with wide scope on making this into formal statements were enunciated by Jean-Marc Fontaine, the resolution of which is called p-adic Hodge theory.

de Rham cohomology

De Rham cohomology solves the problem of finding an algebraic definition of the cohomology groups (singular cohomology)

$H^i(X,\mathbf{C})$

for X a smooth complex variety. These groups are the cohomology of the complex of smooth differential forms on X (with complex number coefficients), as these form a resolution of the constant sheaf $\mathbf{C}$.

The algebraic de Rham cohomology is defined to be the hypercohomology of the complex of algebraic forms (Kähler differentials) on X. The smooth i-forms form an acyclic sheaf, so the hypercohomology of the complex of smooth forms is the same as its cohomology, and the same is true for algebraic sheaves of i-forms over affine varieties, but algebraic sheaves of i-forms over non-affine varieties can have non-vanishing higher cohomology groups, so the hypercohomology can differ from the cohomology of the complex.

For smooth complex varieties Grothendieck (1963) showed that the algebraic de Rham cohomology is isomorphic to the usual smooth de Rham cohomology and therefore (by de Rham's theorem) to the cohomology with complex coefficients. This definition of algebraic de Rham cohomology is available for algebraic varieties over any field k.

Coefficients

If X is a variety over an algebraically closed field of characteristic $p > 0$, then the l-adic cohomology groups for l any prime number other than p give satisfactory cohomology groups of X, with coefficients in the ring $\mathbf{Z}_l$ of l-adic integers. It is not possible in general to find similar cohomology groups with coefficients in the p-adic numbers (or the rationals, or the integers).

The classic reason (due to Serre) is that if X is a supersingular elliptic curve, then its ring of endomorphisms generates a quaternion algebra over $\mathbf{Q}$ that is non-split at p and infinity. If X has a cohomology group over the p-adic integers with the expected dimension 2, the ring of endomorphisms would have a 2-dimensional representation; and this is not possible as it is non-split at p. (A quite subtle point is that if X is a supersingular elliptic curve over the prime field, with p elements, then its crystalline cohomology is a free rank 2 module over the p-adic integers. The argument given does not apply in this case, because some of the endomorphisms of supersingular elliptic curves are only defined over a quadratic extension of the field of order p.)

Grothendieck's crystalline cohomology theory gets around this obstruction because it takes values in the ring of Witt vectors over the ground field. So if the ground field is the algebraic closure of the field of order p, its values are modules over the p-adic completion of the maximal unramified extension of the p-adic integers, a much larger ring containing n-th roots of unity for all n not divisible by p, rather than over the p-adic integers.

Motivation

One idea for defining a Weil cohomology theory of a variety X over a field k of characteristic p is to 'lift' it to a variety X^* over the ring of Witt vectors of k (that gives back X on reduction mod p), then take the de Rham cohomology of this lift. The problem is that it is not at all obvious that this cohomology is independent of the choice of lifting.

The idea of crystalline cohomology in characteristic 0 is to find a direct definition of a cohomology theory as the cohomology of constant sheaves on a suitable site

Inf(X)

over X, called the **infinitesimal site** and then show it is the same as the de Rham cohomology of any lift.

The site Inf(X) is a category whose objects can be thought of as some sort of generalization of the conventional open sets of X. In characteristic 0 its objects are infinitesimal thickenings $U \to T$ of Zariski open subsets U of X. This means that U is the closed subscheme of a scheme T defined by a nilpotent sheaf of ideals on T; for example, Spec(k)→ Spec($k[x]/(x^2)$).

Grothendieck showed that for smooth schemes X over **C**, the cohomology of the sheaf O_X on Inf(X) is the same as the usual (smooth or algebraic) de Rham cohomology.

Crystalline cohomology

In characteristic p the most obvious analogue of the crystalline site defined above in characteristic 0 does not work. The reason is roughly that in order to prove exactness of the de Rham complex, one needs some sort of Poincaré lemma, whose proof in turn uses integration, and integration requires various divided powers, which exist in characteristic 0 but not always in characteristic p. Grothendieck solved this problem by defining objects of the crystalline site of X to be roughly infinitesimal thickenings of Zariski open subsets of X, together with a divided power structure giving the needed divided powers.

We will work over the ring $W_n = W/p^nW$ of Witt vectors of length n over a perfect field k of characteristic p>0. For example, k could be the finite field of order p, and W_n is then the ring $\mathbf{Z}/p^n\mathbf{Z}$. (More generally one can work over a base scheme S which has a fixed sheaf of ideals I with a divided power structure.) If X is a scheme over k, then the **crystalline site of** X **relative to** W_n, denoted Cris(X/W_n), has as its objects pairs $U \to T$ consisting of a closed immersion of a Zariski open subset U of X into some W_n-scheme T defined by a sheaf of ideals J, together with a divided power structure on J compatible with the one on W_n.

Crystalline cohomology of a scheme X over k is defined to be the inverse limit

$$H^i(X/W) = \varprojlim H^i(X/W_n)$$

where

$$H^i(X/W_n) = H^i(Cris(X/W_n), O)$$

is the cohomology of the crystalline site of X/W_n with values in the sheaf of rings $O = O_{X/Wn}$.

A key point of the theory is that the crystalline cohomology of a smooth scheme X over k can often be calculated in terms of the algebraic de Rham cohomology of a proper and smooth lifting of X to a scheme Z over W. There is a canonical isomorphism

$$H^i(X/W) = H^i_{DR}(Z/W) \quad (= H^i(Z, \Omega^*_{Z/W}) = \varprojlim H^i(Z, \Omega^*_{Z/W_n}))$$

of the crystalline cohomology of X with the de Rham cohomology of Z over the formal scheme of W (an inverse limit of the hypercohomology of the complexes of differential forms). Conversely the de Rham cohomology of X can be recovered as the reduction mod p of its crystalline cohomology (after taking higher *Tor*s into account).

Crystals

If X is a scheme over S then the sheaf $O_{X/S}$ is defined by $O_{X/S}(T)$ = coordinate ring of T, where we write T as an abbreviation for an object $U \to T$ of Cris(X/S).

A **crystal** on the site Cris(X/S) is a sheaf F of $O_{X/S}$ modules that is **rigid** in the following sense:

for any map f between objects T, T' of Cris(X/S), the natural map from $f^*F(T)$ to $F(T')$ is an isomorphism.

This is similar to the definition of a quasicoherent sheaf of modules in the Zariski topology.

An example of a crystal is the sheaf $O_{X/S}$.

The term *crystal* attached to the theory, explained in Grothendieck's letter to Tate (1966), was a metaphor inspired by certain properties of algebraic differential equations. These had played a role in p-adic cohomology theories (precursors of the crystalline theory, introduced in various forms by Dwork, Monsky, Washnitzer, Lubkin and Katz) particularly in Dwork's work. Such differential equations can be formulated easily enough by means of the algebraic Koszul connections, but in the p-adic theory the analogue of analytic continuation is more mysterious (since p-adic discs tend to be disjoint rather than overlap). By decree, a *crystal* would have the 'rigidity' and the 'propagation' notable in the case of the analytic continuation of complex analytic functions. (Cf. also the rigid analytic spaces introduced by Tate, in the 1960s, when these matters were actively being debated.)

See also

- Motivic cohomology
- De Rham cohomology

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De Rham cohomology

In mathematics, **de Rham cohomology** (after Georges de Rham) is a tool belonging both to algebraic topology and to differential topology, capable of expressing basic topological information about smooth manifolds in a form particularly adapted to computation and the concrete representation of cohomology classes. It is a cohomology theory based on the existence of differential forms with prescribed properties.

Definition

The **de Rham complex** is the cochain complex of exterior differential forms on some smooth manifold *M*, with the exterior derivative as the differential.

$$0 \to \Omega^0(M) \stackrel{d}{\to} \Omega^1(M) \stackrel{d}{\to} \Omega^2(M) \stackrel{d}{\to} \Omega^3(M) \to \cdots$$

where $\Omega^0(M)$ is the space of smooth functions on *M*, $\Omega^1(M)$ is the space of 1-forms, and so forth. Forms which are the image of other forms under the exterior derivative are called **exact** and forms whose exterior derivative is 0 are called **closed** (see closed and exact differential forms); the relationship $d^2 = 0$ then says that exact forms are closed.

The converse, however, is not in general true; closed forms need not be exact. A simple but significant case is the 1-form of angle measure on the unit circle, written conventionally as dθ. There is no actual function θ defined on the whole circle for which this is true; the increment of 2π in going once round the circle in the positive direction means that we can't take a single-valued θ. We can, however, change the topology by removing just one point.

The idea of de Rham cohomology is to classify the different types of closed forms on a manifold. One performs this classification by saying that two closed forms α and β in $\Omega^k(M)$ are **cohomologous** if they differ by an exact form, that is, if $\alpha - \beta$ is exact. This classification induces an equivalence relation on the space of closed forms in $\Omega^k(M)$. One then defines the k-th **de Rham cohomology group** $H^k_{\mathrm{dR}}(M)$ to be the set of equivalence classes, that is, the set of closed forms in $\Omega^k(M)$ modulo the exact forms.

Note that, for any manifold *M* with *n* connected components

$$H^0_{\mathrm{dR}}(M) \cong \mathbf{R}^n.$$

This follows from the fact that any smooth function on M with zero derivative (i.e. locally constant) is constant on each of the connected components of *M*.

De Rham cohomology computed

One may often find the general de Rham cohomologies of a manifold using the above fact about the zero cohomology and a Mayer–Vietoris sequence. Another useful fact is that the de Rham cohomology is a homotopy invariant. While the computation is not given, the following are the computed de Rham cohomologies for some common topological objects:

The *n*-sphere:

For the *n*-sphere, and also when taken together with a product of open intervals, we have the following. Let $n > 0$, $m \geq 0$, and *I* an open real interval. Then

$$H^k_{\mathrm{dR}}(S^n \times I^m) \simeq \begin{cases} \mathbf{R} & \text{if } k = 0, n, \\ 0 & \text{if } k \neq 0, n. \end{cases}$$

The *n*-torus:

Similarly, allowing $n > 0$ here, we obtain

$$H^k_{\mathrm{dR}}(T^n) \simeq \mathbf{R}^{\binom{n}{k}}.$$

Punctured Euclidean space:

Punctured Euclidean space is simply Euclidean space with the origin removed. For $n > 0$, we have:

$$H^k_{\mathrm{dR}}(\mathbf{R}^n - \{0\}) \simeq \begin{cases} \mathbf{R} & \text{if } k = 0, n-1 \\ 0 & \text{if } k \neq 0, n-1 \end{cases}$$
$$\simeq H^k_{\mathrm{dR}}(S^{n-1}).$$

The Möbius strip, M:

This more-or-less follows from the fact that the Möbius strip may be, loosely speaking, "contracted" to the 1-sphere:

$$H^k_{\mathrm{dR}}(M) \simeq H^k_{\mathrm{dR}}(S^1).$$

De Rham's theorem

Stokes' theorem is an expression of duality between de Rham cohomology and the homology of chains. It says that the pairing of differential forms and chains, via integration, gives a homomorphism from de Rham cohomology $H^k_{\mathrm{dR}}(M)$ to singular cohomology groups $H^k(M; \mathbf{R})$. ***De Rham's theorem**, proved by Georges de Rham in 1931, states that for a smooth manifold* M*, this map is in fact an isomorphism.*

The wedge product endows the direct sum of these groups with a ring structure. A further result of the theorem is that the two cohomology rings are isomorphic (as graded rings), where the analogous product on singular cohomology is the cup product.

Sheaf-theoretic de Rham isomorphism

The de Rham cohomology is isomorphic to the Čech cohomology $H^*(\mathbf{U},F)$, where F is the sheaf of abelian groups determined by $F(U) = \mathbf{R}$ for all connected open sets U in M, and for open sets U and V such that $U \subset V$, the group morphism $\mathrm{res}_{V,U} : F(V) \to F(U)$ is given by the identity map on **R**, and where **U** is a good open cover of M (*i.e.* all the open sets in the open cover **U** are contractible to a point, and all finite intersections of sets in **U** are either empty or contractible to a point).

Stated another way, if M is a compact C^{m+1} manifold of dimension m, then for each $k \leq m$, there is an isomorphism

$$H^k_{\mathrm{dR}}(M, \mathbf{R}) \cong \check{H}^k(M, \mathbf{R})$$

where the left-hand side is the k-th de Rham cohomology group and the right-hand side is the Čech cohomology for the constant sheaf with fibre **R**.

Proof

Let Ω^k denote the sheaf of germs of k-forms on M (with Ω^0 the sheaf of C^{m+1} functions on M). By the Poincaré lemma, the following sequence of sheaves is exact (in the category of sheaves):

$$0 \to \mathbf{R} \to \Omega^0 \xrightarrow{d} \Omega^1 \xrightarrow{d} \Omega^2 \xrightarrow{d} \dots \xrightarrow{d} \Omega^m \to 0.$$

This sequence now breaks up into short exact sequences

$$0 \to d\Omega^{k-1} \xrightarrow{\mathrm{incl}} \Omega^k \xrightarrow{d} d\Omega^k \to 0.$$

Each of these induces a long exact sequence in cohomology. Since the sheaf of C^{m+1} functions on a manifold admits partitions of unity, the sheaf-cohomology $H^i(\Omega^k)$ vanishes for $i > 0$. So the long exact cohomology sequences themselves ultimately separate into a chain of isomorphisms. At one end of the chain is the Čech cohomology and at the other lies the de Rham cohomology.

Related ideas

The de Rham cohomology has inspired many mathematical ideas, including Dolbeault cohomology, Hodge theory, and the Atiyah-Singer index theorem. However, even in more classical contexts, the theorem has inspired a number of developments. Firstly, the Hodge theorem proves that there is an isomorphism between the cohomology consisting of harmonic forms and the de Rham cohomology consisting of closed forms modulo exact forms. This relies on an appropriate definition of **harmonic forms** and of **the Hodge theorem**. For further details see Hodge theory.

Harmonic forms

If M is a compact Riemannian manifold, then each equivalence class in $H^k_{\mathrm{dR}}(M)$ contains exactly one harmonic form. That is, every member ω of a given equivalence class of closed forms can be written as

$$\omega = d\alpha + \gamma$$

where α is some form, and γ is harmonic: $\Delta\gamma=0$.

Any harmonic function on a compact connected Riemannian manifold is a constant. Thus, this particular representative element can be understood to be an extremum (a minimum) of all cohomologously equivalent forms on the manifold. For example, on a 2-torus, one may envision a constant 1-form as one where all of the "hair" is combed neatly in the same direction (and all of the "hair" having the same length). In this case, there are two cohomologically distinct combings; all of the others are linear combinations. In particular, this implies that the 1st Betti number of a two-torus is two. More generally, on an n-dimensional torus T^n, one can consider the various combings of k-forms on the torus. There are n choose k such combings that can be used to form the basis vectors for $H^k_{\mathrm{dR}}(T^n)$; the k-th Betti number for the de Rham cohomology group for the n-torus is thus n choose k.

More precisely, for a differential manifold M, one may equip it with some auxiliary Riemannian metric. Then the Laplacian Δ is defined by

$$\Delta = d\delta + \delta d$$

with d the exterior derivative and δ the codifferential. The Laplacian is a homogeneous (in grading) linear differential operator acting upon the exterior algebra of differential forms: we can look at its action on each component of degree k separately.

If M is compact and oriented, the dimension of the kernel of the Laplacian acting upon the space of k-forms is then equal (by Hodge theory) to that of the de Rham cohomology group in degree k: the Laplacian picks out a unique *harmonic* form in each cohomology class of closed forms. In particular, the space of all harmonic k-forms on M is isomorphic to $H^k(M;\mathbf{R})$. The dimension of each such space is finite, and is given by the k-th Betti number.

Hodge decomposition

Letting δ be the codifferential, one says that a form ω is **co-closed** if $\delta\omega = 0$ and **co-exact** if $\omega = \delta\alpha$ for some form α. The **Hodge decomposition** states that any k-form can be split into three L^2 components:

$$\omega = d\alpha + \delta\beta + \gamma$$

where γ is harmonic: $\Delta\gamma = 0$. This follows by noting that exact and co-exact forms are orthogonal; the orthogonal complement then consists of forms that are both closed and co-closed: that is, of harmonic forms. Here, orthogonality is defined with respect to the L^2 inner product on $\Omega^k(M)$:

$$(\alpha, \beta) = \int_M \alpha \wedge *\beta.$$

A precise definition and proof of the decomposition requires the problem to be formulated on Sobolev spaces. The idea here is that a Sobolev space provides the natural setting for both the idea of square-integrability and the idea of differentiation. This language helps overcome some of the limitations of requiring compact support.

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Algebraic geometry and analytic geometry

In mathematics, **algebraic geometry and analytic geometry** are two closely related subjects. While algebraic geometry studies algebraic varieties, analytic geometry deals with complex manifolds and the more general analytic spaces defined locally by the vanishing of analytic functions of several complex variables. The deep relation between these subjects has numerous applications in which algebraic techniques are applied to analytic spaces and analytic techniques to algebraic varieties.

Background

Algebraic varieties are locally defined as the common zero sets of polynomials and since polynomials over the complex numbers are holomorphic functions, algebraic varieties over **C** can be interpreted as analytic spaces. Similarly, regular morphisms between varieties are interpreted as holomorphic mappings between analytic spaces. Somewhat surprisingly, it is often possible to go the other way, to interpret analytic objects in an algebraic way.

For example, it is easy to prove that the analytic functions from the Riemann sphere to itself are either the rational functions or the identically infinity function (an extension of Liouville's theorem). For if such a function f is nonconstant, then since the set of z where $f(z)$ is infinity is isolated and the Riemann sphere is compact, there are finitely many z with $f(z)$ equal to infinity. Consider the Laurent expansion at all such z and subtract off the singular part: we are left with a function on the Riemann sphere with values in **C**, which by Liouville's theorem is constant. Thus f is a rational function. This fact shows there is no essential difference between the complex projective line as an algebraic variety, or as the Riemann sphere.

Important results

There is a long history of comparison results between algebraic geometry and analytic geometry, beginning in the nineteenth century and still continuing today. Some of the more important advances are listed here in chronological order.

Riemann's existence theorem

Riemann surface theory shows that a compact Riemann surface has enough meromorphic functions on it, making it an algebraic curve. Under the name **Riemann's existence theorem** a deeper result on ramified coverings of a compact Riemann surface was known: such *finite* coverings as topological spaces are classified by permutation representations of the fundamental group of the complement of the ramification points. Since the Riemann surface property is local, such coverings are quite easily seen to be coverings in the complex-analytic sense. It is then possible to conclude that they come from covering maps of algebraic curves — that is, such coverings all come from finite extensions of the function field.

The Lefschetz principle

In the twentieth century, the **Lefschetz principle**, named for Solomon Lefschetz, was cited in algebraic geometry to justify the use of topological techniques for algebraic geometry over any algebraically closed field K of characteristic 0, by treating K as if it were the complex number field. It roughly asserts that true statements in algebraic geometry over $\mathbf{C}$ are true over any algebraically closed field K of characteristic zero. A precise principle and its proof are due to Alfred Tarski and are based in mathematical logic.[1] [2]

This principle permits the carrying over of results obtained using analytic or topological methods for algebraic varieties over $\mathbf{C}$ to other algebraically closed ground fields of characteristic 0.

Chow's theorem

Chow's theorem, proved by W. L. Chow. is an example of the most immediately useful kind of comparison available. It states that an analytic subspace of complex projective space that is closed (in the ordinary topological sense) is an algebraic subvariety. This can be rephrased concisely as "any analytic subspace of complex projective space which is closed in the strong topology is closed in the Zariski topology." This allows quite a free use of complex-analytic methods within the classical parts of algebraic geometry.

Serre's GAGA

Foundations for the many relations between the two theories were put in place during the early part of the 1950s, as part of the business of laying the foundations of algebraic geometry to include, for example, techniques from Hodge theory. The major paper consolidating the theory was *Géometrie Algébrique et Géométrie Analytique* by Serre, now usually referred to as **GAGA**. It proves general results that relate classes of algebraic varieties, regular morphisms and sheaves with classes of analytic spaces, holomorphic mappings and sheaves. It reduces all of these to the comparison of categories of sheaves.

Nowadays the phrase *GAGA-style result* is used for any theorem of comparison, allowing passage between a category of objects from algebraic geometry, and their morphisms, to a well-defined subcategory of analytic geometry objects and holomorphic mappings.

Formal statement of GAGA

1. Let $(X, \mathcal{O}_X)$ be a scheme of finite type over $\mathbf{C}$. Then there is a topological space X^{an} which as a set consists of the closed points of X with a continuous inclusion map λ_X: $X^{an} \to X$. The topology on X^{an} is called the "complex topology" (and is very different from the subspace topology).
2. Suppose φ: $X \to Y$ is a morphism of schemes of locally finite type over $\mathbf{C}$. Then there exists a continuous map φ^{an}: $X^{an} \to Y^{an}$ such $\lambda_Y \circ \varphi^{an} = \varphi \circ \lambda_Y$.
3. There is a sheaf $\mathcal{O}_X^{an}$ on X^{an} such that $(X^{an}, \mathcal{O}_X^{an})$ is a ringed space and λ_X: $X^{an} \to X$ becomes a map of ringed spaces. The space $(X^{an}, \mathcal{O}_X^{an})$ is called the "analytifiction" of $(X, \mathcal{O}_X)$ and is an analytic space. For every φ: $X \to Y$ the map φ^{an} defined above is a mapping of analytic spaces. Furthermore, the map $\varphi \mapsto \varphi^{an}$ maps open immersions into open immersions. If $X = \mathbf{C}[x_1,...,x_n]$ then $X^{an} = \mathbf{C}^n$ and $\mathcal{O}_X^{an}(U)$ for every polydisc U is a suitable quotient of the space of holomorphic functions on U.
4. For every sheaf $\mathcal{F}$ on X (called algebraic sheaf) there is a sheaf $\mathcal{F}^{an}$ on X^{an} (called analytic sheaf) and a map of sheaves of $\mathcal{O}_X$-modules $\lambda_X^* : \mathcal{F} \to (\lambda_X)_* \mathcal{F}^{an}$. The sheaf $\mathcal{F}^{an}$ is defined as $\lambda_X^{-1}\mathcal{F} \otimes_{\lambda_X^{-1}\mathcal{O}_X} \mathcal{O}_X^{an}$. The correspondence $\mathcal{F} \mapsto \mathcal{F}^{an}$ defines an exact functor from the category of sheaves over $(X, \mathcal{O}_X)$ to the category of sheaves of $(X^{an}, \mathcal{O}_X^{an})$.

   The following two statements are the heart of Serre's GAGA theorem (as extended by Grothendieck, Neeman et al.)
5. If f: $X \to Y$ is an arbitrary morphism of schemes of finite type over $\mathbf{C}$ and $\mathcal{F}$ is coherent then the natural map $(f_*\mathcal{F})^{an} \to f_*^{an}\mathcal{F}^{an}$ is injective. If f is proper then this map is an isomorphism. One also has isomorphisms of

all higher direct image sheaves $(R^i f_* \mathcal{F})^{an} \cong R^i f_*^{an} \mathcal{F}^{an}$ in this case.

6. Now assume that X^{an} is hausdorff and compact. If $\mathcal{F}, \mathcal{G}$ are two coherent algebraic sheaves on $(X, \mathcal{O}_X)$ and if $f : \mathcal{F}^{an} \to \mathcal{G}^{an}$ is a map of sheaves of $\mathcal{O}_X^{an}$ modules then there exists a unique map of sheaves of $\mathcal{O}_X$ modules $\varphi : \mathcal{F} \to \mathcal{G}$ with $f = \varphi^{an}$. If $\mathcal{R}$ is a coherent analytic sheaf of $\mathcal{O}_X^{an}$ modules over X^{an} then there exists a coherent algebraic sheaf $\mathcal{F}$ of $\mathcal{O}_X$ -modules and an isomorphism $\mathcal{F}^{an} \cong \mathcal{R}$.

Moishezon manifolds

A **Moishezon manifold** *M* is a compact connected complex manifold such that the field of meromorphic functions on *M* has transcendence degree equal to the complex dimension of *M*. Complex algebraic varieties have this property, but the converse is not (quite) true. The converse is true in the setting of algebraic spaces. In 1967, Boris Moishezon showed that a Moishezon manifold is a projective algebraic variety if and only if it admits a Kähler metric.

Notes

[1] For discussions see A. Seidenberg, *Comments on Lefschetz's Principle*, The American Mathematical Monthly, Vol. 65, No. 9 (Nov., 1958), pp. 685-690; 'Gerhard Frey and Hans-Georg Rück, *The strong Lefschetz principle in algebraic geometry*, Manuscripta Mathematica, Volume 55, Numbers 3-4, September, 1986, pp. 385-401.

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Riemannian manifold

In Riemannian geometry, a **Riemannian manifold** or **Riemannian space** (M,g) is a real differentiable manifold M in which each tangent space is equipped with an inner product g, a **Riemannian metric**, in a manner which varies smoothly from point to point. The metric g is a positive definite symmetric tensor: a metric tensor. In other words, a Riemannian manifold is a differentiable manifold in which the tangent space at each point is a finite-dimensional Euclidean space.

This allows one to define various geometric notions on a Riemannian manifold such as angles, lengths of curves, areas (or volumes), curvature, gradients of functions and divergence of vector fields.

Riemannian manifolds should not be confused with Riemann surfaces, manifolds that locally appear like patches of the complex plane.

The terms are named after German mathematician Bernhard Riemann.

Overview

The tangent bundle of a smooth manifold M assigns to each fixed point of M a vector space called the tangent space, and each tangent space can be equipped with an inner product. If such a collection of inner products on the tangent bundle of a manifold varies smoothly as one traverses the manifold, then concepts that were defined only pointwise at each tangent space can be extended to yield analogous notions over finite regions of the manifold. For example, a smooth curve $\alpha(t)$: $[0, 1] \to M$ has tangent vector $\alpha'(t_0)$ in the tangent space $\mathrm{T}M(t_0)$ at any point $t_0 \in (0, 1)$, and each such vector has length $\|\alpha'(t_0)\|$, where $\|\cdot\|$ denotes the norm induced by the inner product on $\mathrm{T}M(t_0)$. The integral of these lengths gives the length of the curve α:

$$L(\alpha) = \int_0^1 \|\alpha'(t)\| \, \mathrm{d}t.$$

Smoothness of $\alpha(t)$ for t in $[0, 1]$ guarantees that the integral $L(\alpha)$ exists and the length of this curve is defined.

In many instances, in order to pass from a linear-algebraic concept to a differential-geometric one, the smoothness requirement is very important.

Every smooth submanifold of $\mathbf{R}^n$ has an induced Riemannian metric g: the inner product on each tangent space is the restriction of the inner product on $\mathbf{R}^n$. In fact, as follows from the Nash embedding theorem, all Riemannian manifolds can be realized this way. In particular one could *define* Riemannian manifold as a metric space which is isometric to a smooth submanifold of $\mathbf{R}^n$ with the induced intrinsic metric, where isometry here is meant in the sense of preserving the length of curves. This definition might theoretically not be flexible enough, but it is quite useful to build the first geometric intuitions in Riemannian geometry.

Riemannian manifolds as metric spaces

Usually a Riemannian manifold is defined as a smooth manifold with a smooth section of the positive-definite quadratic forms on the tangent bundle. Then one has to work to show that it can be turned to a metric space:

If γ: $[a, b] \to M$ is a continuously differentiable curve in the Riemannian manifold M, then we define its length $L(\gamma)$ in analogy with the example above by

$$L(\gamma) = \int_a^b \|\gamma'(t)\| \, \mathrm{d}t.$$

With this definition of length, every connected Riemannian manifold M becomes a metric space (and even a length metric space) in a natural fashion: the distance $d(x, y)$ between the points x and y of M is defined as

$d(x,y) = \inf\{ \mathrm{L}(\gamma) : \gamma$ is a continuously differentiable curve joining x and $y\}$.

Even though Riemannian manifolds are usually "curved," there is still a notion of "straight line" on them: the geodesics. These are curves which locally join their points along shortest paths.

Assuming the manifold is compact, any two points x and y can be connected with a geodesic whose length is $d(x,y)$. Without compactness, this need not be true. For example, in the punctured plane $\mathbf{R}^2 \setminus \{0\}$, the distance between the points (−1, 0) and (1, 0) is 2, but there is no geodesic realizing this distance.

Properties

In Riemannian manifolds, the notions of geodesic completeness, topological completeness and metric completeness are the same: that each implies the other is the content of the Hopf-Rinow theorem.

Riemannian metrics

Let M be a differentiable manifold of dimension n. A **Riemannian metric** on M is a family of (positive definite) inner products

$$g_p : T_pM \times T_pM \longrightarrow \mathbb{R}, \qquad p \in M$$

such that, for all differentiable vector fields X,Y on M,

$$p \mapsto g_p(X(p), Y(p))$$

defines a smooth function $M \to \mathbf{R}$.

More formally, a Riemannian metric g is a section of the vector bundle

$$S^2T^*M\cdot$$

In a system of local coordinates on the manifold M given by n real-valued functions x^1,x^2, …, x^n, the vector fields

$$\left\{\frac{\partial}{\partial x^1}, \dots, \frac{\partial}{\partial x^n}\right\}$$

give a basis of tangent vectors at each point of M. Relative to this coordinate system, the components of the metric tensor are, at each point p,

$$g_{ij}(p) := g_p\left(\left(\frac{\partial}{\partial x^i}\right)_p, \left(\frac{\partial}{\partial x^j}\right)_p\right).$$

Equivalently, the metric tensor can be written in terms of the dual basis {dx^1, …, dx^n} of the cotangent bundle as

$$g = \sum_{i,j} g_{ij} \mathrm{d}x^i \otimes \mathrm{d}x^j.$$

Endowed with this metric, the differentiable manifold (M,g) is a **Riemannian manifold**.

Examples

- With $\frac{\partial}{\partial x^i}$ identified with $e_i = (0, \dots, 1, \dots, 0)$, the standard metric over an open subset $U \subset \mathbb{R}^n$ is defined by

  $$g_p^{\mathrm{can}} : T_pU \times T_pU \longrightarrow \mathbb{R}, \qquad \left(\sum_i a_i \frac{\partial}{\partial x^i}, \sum_j b_j \frac{\partial}{\partial x^j}\right) \longmapsto \sum_i a_i b_i.$$

  Then g is a Riemannian metric, and

  $$g_{ij}^{\mathrm{can}} = \langle e_i, e_j \rangle = \delta_{ij}.$$

  Equipped with this metric, $\mathbf{R}^n$ is called **Euclidean space** of dimension n and g_{ij}^{can} is called the **Euclidean metric**.

- Let (M,g) be a Riemannian manifold and $N \subset M$ be a submanifold of M. Then the restriction of g to vectors tangent along N defines a Riemannian metric over N.
- More generally, let $f: M^n \to N^{n+k}$ be an immersion. Then, if N has a Riemannian metric, f induces a Riemannian metric on M via pullback:

$$g_p^M : T_pM \times T_pM \longrightarrow \mathbb{R},$$
$$(u,v) \longmapsto g_p^M(u,v) := g_{f(p)}^N(T_pf(u), T_pf(v)).$$

This is then a metric; the positive definiteness follows of the injectivity of the differential of an immersion.

- Let (M,g^M) be a Riemannian manifold, $h: M^{n+k} \to N^k$ be a differentiable map and $q \in N$ be a regular value of h (the differential $dh(p)$ is surjective for all $p \in h^{-1}(q)$). Then $h^{-1}(q) \subset M$ is a submanifold of M of dimension n. Thus $h^{-1}(q)$ carries the Riemannian metric induced by inclusion.
- In particular, consider the following map :

$$h : \mathbb{R}^n \longrightarrow \mathbb{R}, \qquad (x^1, \ldots, x^n) \longmapsto \sum_{i=1}^n (x^i)^2 - 1.$$

Then, 0 is a regular value of h and

$$h^{-1}(0) = \{x \in \mathbb{R}^n | \sum_{i=1}^n (x^i)^2 = 1\} = S^{n-1}$$

is the unit sphere $S^{n-1} \subset \mathbb{R}^n$. The metric induced from $\mathbb{R}^n$ on S^{n-1} is called the **canonical metric** of S^{n-1}.

- Let M_1 and M_2 be two Riemannian manifolds and consider the cartesian product $M_1 \times M_2$ with the product structure. Furthermore, let $\pi_1 : M_1 \times M_2 \to M_1$ and $\pi_2 : M_1 \times M_2 \to M_2$ be the natural projections. For $(p,q) \in M_1 \times M_2$, a Riemannian metric on $M_1 \times M_2$ can be introduced as follows :

$$g_{(p,q)}^{M_1 \times M_2} : T_{(p,q)}(M_1 \times M_2) \times T_{(p,q)}(M_1 \times M_2) \longrightarrow \mathbb{R},$$
$$(u,v) \longmapsto g_p^{M_1}(T_{(p,q)}\pi_1(u), T_{(p,q)}\pi_1(v)) + g_q^{M_2}(T_{(p,q)}\pi_2(u), T_{(p,q)}\pi_2(v)).$$

The identification

$$T_{(p,q)}(M_1 \times M_2) \cong T_pM_1 \oplus T_qM_2$$

allows us to conclude that this defines a metric on the product space.

The torus $S^1 \times \ldots \times S^1 = T^n$ possesses for example a Riemannian structure obtained by choosing the induced Riemannian metric from $\mathbb{R}^2$ on the circle $S^1 \subset \mathbb{R}^2$ and then taking the product metric. The torus T^n endowed with this metric is called the flat torus.

- Let g_0, g_1 be two metrics on M. Then,

$$\tilde{g} := \lambda g_0 + (1-\lambda) g_1, \qquad \lambda \in [0,1],$$

is also a metric on M.

The pullback metric

If $f{:}M{\to}N$ is a differentiable map and (N,g^N) a Riemannian manifold, then the pullback of g^N along f is a quadratic form on the tangent space of M. The pullback is the quadratic form f^*g^N on TM defined for $v, w \in T_pM$ by

$$(f^*g^N)(v,w) = g^N(df(v), df(w)) .$$

where $df(v)$ is the pushforward of v by f.

The quadratic form f^*g^N is in general only a semi definite form because df can have a kernel. If f is a diffeomorphism, or more generally an immersion, then it defines a Riemannian metric on M, the pullback metric. In particular, every embedded smooth submanifold inherits a metric from being embedded in a Riemannian manifold, and every covering space inherits a metric from covering a Riemannian manifold.

Existence of a metric

Every paracompact differentiable manifold admits a Riemannian metric. To prove this result, let M be a manifold and $\{(U_\alpha, \varphi(U_\alpha))|\alpha \in I\}$ a locally finite atlas of open subsets U of M and diffeomorphisms onto open subsets of $\mathbf{R}^n$

$$\phi : U_\alpha \to \phi(U_\alpha) \subseteq \mathbb{R}^n.$$

Let τ_α be a differentiable partition of unity subordinate to the given atlas. Then define the metric g on M by

$$g := \sum_\beta \tau_\beta \cdot \tilde{g}_\beta, \qquad \text{with} \qquad \tilde{g}_\beta := \tilde{\phi}^*_\beta g^{\text{can}}.$$

where g^{can} is the Euclidean metric. This is readily seen to be a metric on M.

Isometries

Let (M, g^M) and (N, g^N) be two Riemannian manifolds, and $f : M \to N$ be a diffeomorphism. Then, f is called an **isometry**, if

$$g^M = f^*g^N ,$$

or pointwise

$$g^M_p(u,v) = g^N_{f(p)}(T_pf(u), T_pf(v)) \qquad \forall p \in M, \forall u,v \in T_pM.$$

Moreover, a differentiable mapping $f : M \to N$ is called a **local isometry** at $p \in M$ if there is a neighbourhood $U \subset M$, $U \ni p$, such that $f : U \to f(U)$ is a diffeomorphism satisfying the previous relation.

Riemannian manifolds as metric spaces

A connected Riemannian manifold carries the structure of a metric space whose distance function is the arclength of a minimizing geodesic.

Specifically, let (M,g) be a connected Riemannian manifold. Let $c : [a,b] \to M$ be a parametrized curve in M, which is differentiable with velocity vector c'. The length of c is defined as

$$L_a^b(c) := \int_a^b \sqrt{g(c'(t), c'(t))}\,\mathrm{d}t = \int_a^b \|c'(t)\|\,\mathrm{d}t.$$

By change of variables, the arclength is independent of the chosen parametrization. In particular, a curve $[a,b] \to M$ can be parametrized by its arc length. A curve is parametrized by arclength if and only if $\|c'(t)\| = 1$ for all $t \in [a,b]$.

The distance function $d : M{\times}M \to [0,\infty)$ is defined by

$$d(p,q) = \inf L(\gamma)$$

where the infimum extends over all differentiable curves γ beginning at $p{\in}M$ and ending at $q{\in}M$.

This function d satisfies the properties of a distance function for a metric space. The only property which is not completely straightforward is to show that $d(p,q)=0$ implies that $p=q$. For this property, one can use a normal coordinate system, which also allows one to show that the topology induced by d is the same as the original topology on M.

Diameter

The **diameter** of a Riemannian manifold M is defined by

$$\operatorname{diam}(M) := \sup_{p,q\in M} d(p,q) \in \mathbb{R}_{\geq 0} \cup \{+\infty\}.$$

The diameter is invariant under global isometries. Furthermore, the Heine-Borel property holds for (finite-dimensional) Riemannian manifolds: M is compact if and only if it is complete and has finite diameter.

Geodesic completeness

A Riemannian manifold M is **geodesically complete** if for all $p \in M$, the exponential map $\exp_p$ is defined for all $v \in T_pM$, i.e. if any geodesic $\gamma(t)$ starting from p is defined for all values of the parameter $t \in \mathbb{R}$. The Hopf-Rinow theorem asserts that M is geodesically complete if and only if it is complete as a metric space.

If M is complete, then M is non-extendable in the sense that it is not isometric to an open proper submanifold of any other Riemannian manifold. The converse is not true, however: there exist non-extendable manifolds which are not complete.

See also

- Riemannian geometry
- Finsler manifold
- sub-Riemannian manifold
- pseudo-Riemannian manifold
- Metric tensor
- Hermitian manifold
- Space (mathematics)

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External links

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[2] http://eom.springer.de/R/r082180.htm

List of complex analysis topics

This is a **list of complex analysis topics**, by Wikipedia page.

Local theory

- Holomorphic function
- Antiholomorphic function
- Cauchy-Riemann equations
- Conformal mapping
- Power series
- Radius of convergence
- Laurent series
- Meromorphic function
- Entire function
- Pole (complex analysis)
- Zero (complex analysis)
- Residue (complex analysis)
- Isolated singularity
- Removable singularity
- Essential singularity
- Branch point
- Principal branch
- Weierstrass-Casorati theorem
- Landau's constants
- Holomorphic functions are analytic
- Schwarzian derivative
- Analytic capacity
- Disk algebra

Growth

- Bieberbach conjecture
- Borel-Carathéodory theorem
- Hadamard three-circle theorem
- Hardy space
  - Hardy's theorem
  - Progressive function
  - Corona theorem
- Maximum modulus principle
- Nevanlinna theory
- Picard's theorem

- Paley-Wiener theorem
- Value distribution theory of holomorphic functions

Contour integrals

- Line integral
- Cauchy integral theorem
- Cauchy's integral formula
- Residue theorem
- Liouville's theorem (complex analysis)
- Examples of contour integration
- Fundamental theorem of algebra
- Simply connected
- Winding number
    - Principle of the argument
    - Rouché's theorem
- Bromwich integral
- Morera's theorem
- Mellin transform
- Kramers–Kronig relation

Special functions

- Exponential function
- Beta function
- Gamma function
- Riemann zeta function
    - Riemann hypothesis
    - Generalized Riemann hypothesis
- Elliptic function
    - Half-period ratio
    - Jacobi's elliptic functions
    - Weierstrass's elliptic functions
    - Theta function
- Elliptic modular function
- J-function
- Modular function
- Modular form

Riemann surfaces

- Analytic continuation
- Riemann sphere
- Riemann surface
- Riemann mapping theorem
- Carathéodory's theorem (conformal mapping)
- Riemann-Roch theorem

Other

- Antiderivative (complex analysis)
- Bôcher's theorem
- Cayley transform
- Complex differential equation
- Harmonic conjugate
- Method of steepest descent
- Montel's theorem
- Periodic points of complex quadratic mappings
- Pick matrix
- Runge approximation theorem
- Schwarz lemma
- Weierstrass factorization theorem
- Mittag-Leffler's theorem

Several complex variables

- Analytization trick
- Biholomorphy
- Cartan's theorems A and B
- Cousin problems
- Edge-of-the-wedge theorem
- Several complex variables

History (needs work)

- Augustin Louis Cauchy
- Jacques Hadamard
- Kiyoshi Oka
- Bernhard Riemann
- Karl Weierstrass

Algebraic Topology and Groupoids

Algebraic topology

Algebraic topology is a branch of mathematics which uses tools from abstract algebra to study topological spaces. The basic goal is to find algebraic invariants that classify topological spaces up to homeomorphism, though usually most classify up to homotopy equivalence.

Although algebraic topology primarily uses algebra to study topological problems, using topology to solve algebraic problems is sometimes also possible. Algebraic topology, for example, allows for a convenient proof that any subgroup of a free group is again a free group.

The method of algebraic invariants

An older name for the subject was combinatorial topology, implying an emphasis on how a space X was constructed from simpler ones (the modern standard tool for such construction is the CW-complex). The basic method now applied in algebraic topology is to investigate spaces via algebraic invariants by mapping them, for example, to groups which have a great deal of manageable structure in a way that respects the relation of homeomorphism (or more general homotopy) of spaces. This allows one to recast statements about topological spaces into statements about groups, which are often easier to prove.

Two major ways in which this can be done are through fundamental groups, or more generally homotopy theory, and through homology and cohomology groups. The fundamental groups give us basic information about the structure of a topological space, but they are often nonabelian and can be difficult to work with. The fundamental group of a (finite) simplicial complex does have a finite presentation.

Homology and cohomology groups, on the other hand, are abelian and in many important cases finitely generated. Finitely generated abelian groups are completely classified and are particularly easy to work with.

Setting in category theory

In general, all constructions of algebraic topology are functorial; the notions of category, functor and natural transformation originated here. Fundamental groups and homology and cohomology groups are not only *invariants* of the underlying topological space, in the sense that two topological spaces which are homeomorphic have the same associated groups, but their associated morphisms also correspond — a continuous mapping of spaces induces a group homomorphism on the associated groups, and these homomorphisms can be used to show non-existence (or, much more deeply, existence) of mappings.

Results on homology

Several useful results follow immediately from working with finitely generated abelian groups. The free rank of the n-th homology group of a simplicial complex is equal to the n-th Betti number, so one can use the homology groups of a simplicial complex to calculate its Euler-Poincaré characteristic. As another example, the top-dimensional integral homology group of a closed manifold detects orientability: this group is isomorphic to either the integers or 0, according as the manifold is orientable or not. Thus, a great deal of topological information is encoded in the homology of a given topological space.

Beyond simplicial homology, which is defined only for simplicial complexes, one can use the differential structure of smooth manifolds via de Rham cohomology, or Čech or sheaf cohomology to investigate the solvability of differential equations defined on the manifold in question. De Rham showed that all of these approaches were interrelated and that, for a closed, oriented manifold, the Betti numbers derived through simplicial homology were the same Betti numbers as those derived through de Rham cohomology. This was extended in the 1950s, when Eilenberg and Steenrod generalized this approach. They defined homology and cohomology as functors equipped with natural transformations subject to certain axioms (e.g., a weak equivalence of spaces passes to an isomorphism of homology groups), verified that all existing (co)homology theories satisfied these axioms, and then proved that such an axiomatization uniquely characterized the theory.

A new approach uses a functor from filtered spaces to crossed complexes defined directly and homotopically using relative homotopy groups; a higher homotopy van Kampen theorem proved for this functor enables basic results in algebraic topology, especially on the border between homology and homotopy, to be obtained without using singular homology or simplicial approximation. This approach is also called nonabelian algebraic topology, and generalises to higher dimensions ideas coming from the fundamental group.

Applications of algebraic topology

Classic applications of algebraic topology include:

- The Brouwer fixed point theorem: every continuous map from the unit n-disk to itself has a fixed point.
- The n-sphere admits a nowhere-vanishing continuous unit vector field if and only if n is odd. (For $n = 2$, this is sometimes called the "hairy ball theorem".)
- The Borsuk–Ulam theorem: any continuous map from the n-sphere to Euclidean n-space identifies at least one pair of antipodal points.
- Any subgroup of a free group is free. This result is quite interesting, because the statement is purely algebraic yet the simplest proof is topological. Namely, any free group G may be realized as the fundamental group of a graph X. The main theorem on covering spaces tells us that every subgroup H of G is the fundamental group of some covering space Y of X; but every such Y is again a graph. Therefore its fundamental group H is free.

On the other hand this type of application is also handled by the use of covering morphisms of groupoids, and that technique has yielded subgroup theorems not yet proved by methods of algebraic topology (see the book by Higgins listed under groupoids).

- Topological combinatorics

Notable algebraic topologists

- Frank Adams
- Karol Borsuk
- Luitzen Egbertus Jan Brouwer
- William Browder
- Ronald Brown (mathematician)
- Nicolas Bourbaki
- Henri Cartan
- Hermann Künneth
- Samuel Eilenberg
- Hans Freudenthal
- Peter Freyd
- Alexander Grothendieck
- Friedrich Hirzebruch
- Heinz Hopf
- Michael J. Hopkins

- Witold Hurewicz
- Egbert van Kampen
- Saunders Mac Lane
- Jean Leray
- Mark Mahowald
- J. Peter May
- John Coleman Moore
- Sergei Novikov
- Lev Pontryagin
- Mikhail Postnikov
- Daniel Quillen
- Jean-Pierre Serre
- Stephen Smale
- Norman Steenrod
- Dennis Sullivan
- René Thom
- Leopold Vietoris
- Hassler Whitney
- J. H. C. Whitehead
- Brandon Blaha

Important theorems in algebraic topology

- Borsuk-Ulam theorem
- Brouwer fixed point theorem
- Cellular approximation theorem
- Eilenberg–Zilber theorem
- Freudenthal suspension theorem
- Hurewicz theorem
- Kunneth theorem
- Poincaré duality theorem
- Universal coefficient theorem
- Van Kampen's theorem
- Generalized van Kampen's theorems [1]
- Higher homotopy, generalized van Kampen's theorem[2]
- Whitehead's theorem

Notes

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Groupoid

In mathematics, especially in category theory and homotopy theory, a **groupoid** (less often **Brandt groupoid** or **virtual group**) generalises the notion of group in several equivalent ways. A groupoid can be seen as a:

- *Group* with a partial function replacing the binary operation;
- *Category* in which every morphism is invertible. A category of this sort can be viewed as augmented with a unary operation, called *inverse* by analogy with group theory.

Special cases include:

- *Setoids*, that is: sets which come with an equivalence relation;
- *G-sets*, sets equipped with an action of a group *G*.

Groupoids are often used to reason about geometrical objects such as manifolds. Heinrich Brandt introduced groupoids implicitly via Brandt semigroups in 1926.[1]

Definitions

Algebraic

A groupoid is a set G with a unary operation $^{-1} : G \to G,$ and a partial function $* : G \times G \to G.$ * is not a binary operation because it is not necessarily defined for all possible pairs of G-elements. The precise conditions under which * is defined are not articulated here and vary by situation.
$*$ and $^{-1}$ have the following axiomatic properties. Let a, b, and c be elements of G. Then:

- *Associativity*: If $a * b$ and $b * c$ are defined, then $(a * b) * c$ and $a * (b * c)$ are defined and equal. Conversely, if either of these last two expressions is defined, then so is the other (and again they are equal).
- *Inverse*: $a^{-1} * a$ and $a * a^{-1}$ are always defined.
- *Identity*: If $a * b$ is defined, then $a * b * b^{-1} = a$, and $a^{-1} * a * b = b$. (The previous two axioms already show that these expressions are defined and unambiguous.)

In short:

- $(a * b) * c = a * (b * c)$;
- $(a * b) * b^{-1} = a$;
- $a^{-1} * (a * b) = b$.

From these axioms, two easy and convenient theorems follow:

- $(a^{-1})^{-1} = a$;
- If $a * b$ is defined, then $(a * b)^{-1} = b^{-1} * a^{-1}$.

Category theoretic

A groupoid is a small category in which every morphism is an isomorphism, and hence invertible. More precisely, a groupoid G is:

- A set G_0 of *objects*;
- For each pair of objects x and y in G_0, there exists a (possibly empty) set $G(x,y)$ of *morphisms* (or *arrows*) from x to y. We write $f : x \to y$ to indicate that f is an element of $G(x,y)$.

The objects and morphisms have the properties:

- For every object x, there exists the element id_x of $G(x,x)$;
- For each triple of objects x, y, and z, there exists the function $\mathrm{comp}_{x,y,z} : G(x,y) \times G(y,z) \to G(x,z)$. We write gf for $\mathrm{comp}_{x,y,z}(f, g)$, where $f \in G(x,y)$, and $g \in G(y,z)$;

- There exists the function $\mathrm{inv}_{x,y} : G(x,y) \to G(y,x)$.

Moreover, if $f : x \to y$, $g : y \to z$, and $h : z \to w$, then:

- $f\mathrm{id}_x = f$ and $\mathrm{id}_y f = f$;
- $(hg)f = h(gf)$;
- $f\mathrm{inv}(f) = \mathrm{id}_y$ and $\mathrm{inv}(f)f = \mathrm{id}_x$.

If f is an element of $G(x,y)$ then x is called the **source** of f, written $s(f)$, and y the **target** of f (written $t(f)$).

Comparing the definitions

The algebraic and category-theoretic definitions are equivalent, as follows. Given a groupoid in the category-theoretic sense, let G be the disjoint union of all of the sets $G(x,y)$ (i.e. the sets of morphisms from x to y). Then comp and inv become partially defined operations on G, and inv will in fact be defined everywhere; so we define $*$ to be comp and -1 to be inv. Thus we have a groupoid in the algebraic sense. Explicit reference to G_0 (and hence to id) can be dropped.

Conversely, given a groupoid G in the algebraic sense, with typical element f, let G_0 be the set of all elements of the form $f*f^{-1}$. In other words, the objects are identified with the identity morphisms, so that id_x is just x. Let $G(x,y)$ be the set of all elements f such that $y\,f\,x$ is defined. Then $^{-1}$ and $*$ break up into several functions on the various $G(x,y)$, which may be called inv and comp, respectively.

Sets in the definitions above may be replaced with classes, as is generally the case in category theory.

Vertex groups

Given a groupoid G, the **vertex groups** or **isotropy group** in G are the subsets of the form $G(x,x)$, where x is any object of G. It follows easily from the axioms above that these are indeed groups, as every pair of elements is composable and inverses are in the same vertex group.

Category of groupoids

The category whose objects are groupoids and whose morphisms are groupoid homomorphisms is called the **groupoid category**, or the **category of groupoids**.

Examples

Linear algebra

Given a field K, the corresponding **general linear groupoid** $GL_*(K)$ consists of all invertible matrices whose entries range over K. Matrix multiplication interprets composition. If $G = GL_*(K)$, then the set of natural numbers is a proper subset of G_0, since for each natural number n, there is a corresponding identity matrix of dimension n. $G(m,n)$ is empty unless $m=n$, in which case it is the set of all nxn invertible matrices.

Topology

Given a topological space X, let G_0 be the set X. The morphisms from the point p to the point q are equivalence classes of continuous paths from p to q, with two paths being equivalent if they are homotopic. Two such morphisms are composed by first following the first path, then the second; the homotopy equivalence guarantees that this composition is associative. This groupoid is called the fundamental groupoid of X, denoted $\pi_1(X)$. The usual fundamental group $\pi_1(X,x)$ is then the vertex group for the point x.

An important extension of this idea is to consider the fundamental groupoid $\pi_1(X,A)$ where A is a set of "base points" and a subset of X. Here, one considers only paths whose endpoints belong to A. $\pi_1(X,A)$ is a sub-groupoid of $\pi_1(X)$. The set A may be chosen according to the geometry of the situation at hand.

Equivalence relation

If X is a set with an equivalence relation denoted by infix $\sim$, then a groupoid "representing" this equivalence relation can be formed as follows:

- The objects of the groupoid are the elements of X;
- For any two elements x and y in X, there is a single morphism from x to y if and only if $x \sim y$.

Group action

If the group G acts on the set X, then we can form the **action groupoid** representing this group action as follows:

- The objects are the elements of X;
- For any two elements x and y in X, there is a morphism from x to y corresponding to every element g of G such that $gx = y$;
- Composition of morphisms interprets the binary operation of G.

More explicitly, the *action groupoid* is the set $G \times X$ with source and target maps $s(g,x) = x$ and $t(g,x) = gx$. It is often denoted $G \ltimes X$ (or $X \rtimes G$). Multiplication (or composition) in the groupoid is then $(h,y)(g,x) = (hg,x)$ which is defined provided $y=gx$.
For x in X, the vertex group consists of those (g,x) with $gx = x$, which is just the isotropy subgroup at x for the given action (which is why vertex groups are also called isotropy groups).

Another way to describe G-sets is the functor category $[\mathrm{Gr}, \mathsf{Set}]$, where Gr is the groupoid (category) with one element and isomorphic to the group G. Indeed, every functor F of this category defines a set $X=F\,(\mathrm{Gr})$ and for every g in G (i.e. for every morphism in Gr) induces a bijection $F_g : X \to X$. The categorical structure of the functor F assures us that F defines a G-action on the set X. The (unique) representable functor $F : \mathrm{Gr} \to \mathsf{Set}$ is the Cayley representation of G. In fact, this functor is isomorphic to $\mathrm{Hom}(\mathrm{Gr}, -)$ and so sends $\mathrm{ob}(\mathrm{Gr})$ to the set $\mathrm{Hom}(\mathrm{Gr}, \mathrm{Gr})$ which is by definition the "set" G and the morphism g of Gr (i.e. the element g of G) to the permutation F_g of the set G. We deduce from the Yoneda embedding that the group G is isomorphic to the group $\{F_g \mid g \in G\}$, a subgroup of the group of permutations of G.

Fifteen puzzle

The symmetries of the fifteen puzzle form a groupoid (not a group, as not all moves can be composed).[2] [3] This groupoid acts on configurations.

Relation to groups

Group-like structures				
	Totality	**Associativity**	**Identity**	**Inverses**
Group	Yes	Yes	Yes	Yes
Monoid	Yes	Yes	Yes	No
Semigroup	Yes	Yes	No	No
Loop	Yes	No	Yes	Yes
Quasigroup	Yes	No	No	Yes
Magma	Yes	No	No	No
Groupoid	No	Yes	Yes	Yes
Category	No	Yes	Yes	No

If a groupoid has only one object, then the set of its morphisms forms a group. Using the algebraic definition, such a groupoid is literally just a group. Many concepts of group theory generalize to groupoids, with the notion of functor replacing that of group homomorphism.

If x is an object of the groupoid G, then the set of all morphisms from x to x forms a group $G(x)$. If there is a morphism f from x to y, then the groups $G(x)$ and $G(y)$ are isomorphic, with an isomorphism given by the mapping $g \to fgf^{-1}$.

Every connected groupoid (that is, one in which any two objects are connected by at least one morphism) is isomorphic to a groupoid of the following form. Pick a group G and a set (or class) X. Let the objects of the groupoid be the elements of X. For elements x and y of X, let the set of morphisms from x to y be G. Composition of morphisms is the group operation of G. If the groupoid is not connected, then it is isomorphic to a disjoint union of groupoids of the above type (possibly with different groups G for each connected component). Thus any groupoid may be given (up to isomorphism) by a set of ordered pairs (X,G).

Note that the isomorphism described above is not unique, and there is no natural choice. Choosing such an isomorphism for a connected groupoid essentially amounts to picking one object x_0, a group isomorphism h from $G(x_0)$ to G, and for each x other than x_0, a morphism in G from x_0 to x.

In category-theoretic terms, each connected component of a groupoid is equivalent (but not isomorphic) to a groupoid with a single object, that is, a single group. Thus any groupoid is equivalent to a multiset of unrelated groups. In other words, for equivalence instead of isomorphism, one need not specify the sets X, only the groups G.

Consider the examples in the previous section. The general linear groupoid is both equivalent and isomorphic to the disjoint union of the various general linear groups $GL_n(F)$. On the other hand:

- The fundamental groupoid of X is equivalent to the collection of the fundamental groups of each path-connected component of X, but an isomorphism requires specifying the set of points in each component;
- The set X with the equivalence relation $\sim$ is equivalent (as a groupoid) to one copy of the trivial group for each equivalence class, but an isomorphism requires specifying what each equivalence class is:
- The set X equipped with an action of the group G is equivalent (as a groupoid) to one copy of G for each orbit of the action, but an isomorphism requires specifying what set each orbit is.

The collapse of a groupoid into a mere collection of groups loses some information, even from a category-theoretic point of view, because it is not natural. Thus when groupoids arise in terms of other structures, as in the above examples, it can be helpful to maintain the full groupoid. Otherwise, one must choose a way to view each $G(x)$ in terms of a single group, and this choice can be arbitrary. In our example from topology, you would have to make a coherent choice of paths (or equivalence classes of paths) from each point p to each point q in the same path-connected component.

As a more illuminating example, the classification of groupoids with one endomorphism does not reduce to purely group theoretic considerations. This is analogous to the fact that the classification of vector spaces with one endomorphism is nontrivial.

Morphisms of groupoids come in more kinds than those of groups: we have, for example, fibrations, covering morphisms, universal morphisms, and quotient morphisms. Thus a subgroup H of a group G yields an action of G on the set of cosets of H in G and hence a covering morphism p from, say, K to G, where K is a groupoid with vertex groups isomorphic to H. In this way, presentations of the group G can be "lifted" to presentations of the groupoid K, and this is a useful way of obtaining information about presentations of the subgroup H. For further information, see the books by Higgins and by Brown in the References.

Another useful fact is that the category of groupoids, unlike that of groups, is cartesian closed.

Lie groupoids and Lie algebroids

When studying geometrical objects, the arising groupoids often carry some differentiable structure, turning them into Lie groupoids. These can be studied in terms of Lie algebroids, in analogy to the relation between Lie groups and Lie algebras.

Notes

[1] Brandt semigroup (http://eom.springer.de/b/b017600.htm) in Springer Encyclopaedia of Mathematics - ISBN 1-4020-0609-8
[2] The 15-puzzle groupoid (1) (http://www.neverendingbooks.org/index.php/the-15-puzzle-groupoid-1.html), Never Ending Books
[3] The 15-puzzle groupoid (2) (http://www.neverendingbooks.org/index.php/the-15-puzzle-groupoid-2.html), Never Ending Books

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- Weinstein, Alan, " Groupoids: unifying internal and external symmetry -- A tour though some examples. (http://www.ams.org/notices/199607/weinstein.pdf)" Also available in Postscript. (http://math.berkeley.edu/~alanw/Groupoids.ps), Notices of AMS, July 1996, pp. 744-752.

Galois group

In mathematics, more specifically in the area of modern algebra known as Galois theory, the **Galois group** of a certain type of field extension is a specific group associated with the field extension. The study of field extensions (and polynomials which give rise to them) via Galois groups is called Galois theory, so named in honor of Évariste Galois who first discovered them.

For a more elementary discussion of Galois groups in terms of permutation groups, see the article on Galois theory.

Definition

Suppose that E is an extension of the field F (written as E/F and read E over F). Consider the set of all automorphisms of E/F (that is, isomorphisms α from E to itself such that $\alpha(x) = x$ for every x in F). This set of automorphisms with the operation of function composition forms a group, sometimes denoted by $\mathrm{Aut}(E/F)$.

If E/F is a Galois extension, then $\mathrm{Aut}(E/F)$ is called the **Galois group of (the extension) *E* over *F***, and is usually denoted by $\mathrm{Gal}(E/F)$.[1]

Examples

In the following examples F is a field, and $\mathbf{C}$, $\mathbf{R}$, $\mathbf{Q}$ are the fields of complex, real, and rational numbers, respectively. The notation $F(a)$ indicates the field extension obtained by adjoining an element a to the field F.

- $\mathrm{Gal}(F/F)$ is the trivial group that has a single element, namely the identity automorphism.
- $\mathrm{Gal}(\mathbf{C}/\mathbf{R})$ has two elements, the identity automorphism and the complex conjugation automorphism.
- $\mathrm{Aut}(\mathbf{R}/\mathbf{Q})$ is trivial. Indeed it can be shown that any $\mathbf{Q}$-automorphism must preserve the ordering of the real numbers and hence must be the identity.
- $\mathrm{Aut}(\mathbf{C}/\mathbf{Q})$ is an infinite group.
- $\mathrm{Gal}(\mathbf{Q}(\sqrt{2})/\mathbf{Q})$ has two elements, the identity automorphism and the automorphism which exchanges $\sqrt{2}$ and $-\sqrt{2}$.
- Consider the field $K = \mathbf{Q}(\sqrt[3]{2})$. The group $\mathrm{Aut}(K/\mathbf{Q})$ contains only the identity automorphism. This is because K is not a normal extension, since the other two cube roots of 2 (both complex) are missing from the extension — in other words K is not a splitting field.
- Consider now $L = \mathbf{Q}(\sqrt[3]{2}, \omega)$, where ω is a primitive third root of unity. The group $\mathrm{Gal}(L/\mathbf{Q})$ is isomorphic to S_3, the dihedral group of order 6, and L is in fact the splitting field of $x^3 - 2$ over $\mathbf{Q}$.
- If q is a prime power, and if $F = \mathbf{GF}(q)$ and $E = \mathbf{GF}(q^n)$ denote the Galois fields of order q and q^n respectively, then $\mathrm{Gal}(E/F)$ is cyclic of order n.

Properties

The significance of an extension being Galois is that it obeys the fundamental theorem of Galois theory: the closed (with respect to the Krull topology below) subgroups of the Galois group correspond to the intermediate fields of the field extension.

If E/F is a Galois extension, then $\mathrm{Gal}(E/F)$ can be given a topology, called the Krull topology, that makes it into a profinite group.

Notes

[1] Some authors refer to Aut(*E*/*F*) as the Galois group for arbitrary extensions *E*/*F* and use the corresponding notation, e.g. Jacobson 2009.

References

- Jacobson, Nathan (2009) [1985], *Basic algebra I* (Second ed.), Dover Publications, ISBN 978-0-486-47189-1
- Lang, Serge (2002), *Algebra*, Graduate Texts in Mathematics, **211** (Revised third ed.), New York: Springer-Verlag, MR1878556, ISBN 978-0-387-95385-4

External links

- Galois Groups (http://www.mathpages.com/home/kmath290/kmath290.htm) at MathPages

Grothendieck group

In mathematics, the **Grothendieck group** construction in abstract algebra constructs an abelian group from a commutative monoid in the best possible way. It takes its name from the more general construction in category theory, introduced by Alexander Grothendieck in his fundamental work of the mid-1950s that resulted in the development of K-theory. The Grothendieck group is denoted by K or K_0.

Universal property

In its simplest form, the Grothendieck group of a commutative monoid is the universal way of making that monoid into an abelian group. Let M be a commutative monoid. Its Grothendieck group N should have the following universal property: There exists a monoid homomorphism

$$i{:}M\to N$$

such that for any monoid homomorphism

$$f{:}M\to A$$

from the commutative monoid M to an abelian group A, there is a unique group homomorphism

$$g{:}N\to A$$

such that

$$f=gi.$$

In the language of category theory, the functor that sends a commutative monoid M to its Grothendieck group N is left adjoint to the forgetful functor from the category of abelian groups to the category of commutative monoids.

Explicit construction

To construct the Grothendieck group of a commutative monoid M, one forms the Cartesian product

$$M\times M.$$

The two coordinates are meant to represent a positive part and a negative part:

$$(m, n)$$

is meant to correspond to

$$m - n.$$

Addition is defined coordinate-wise:

$$(m_1, m_2) + (n_1, n_2) = (m_1 + n_1, m_2 + n_2).$$

Next we define an equivalence relation on $M \times M$. We say that (m_1, m_2) is equivalent to (n_1, n_2) if, for some element k of M, $m_1 + n_2 + k = m_2 + n_1 + k$. It is easy to check that the addition operation is compatible with the equivalence relation. The identity element is now any element of the form (m, m), and the inverse of (m_1, m_2) is (m_2, m_1).

In this form, the Grothendieck group is the fundamental construction of K-theory. The group $K_0(M)$ of a manifold M is defined to be the Grothendieck group of the commutative monoid of all isomorphism classes of vector bundles of finite rank on M with the monoid operation given by direct sum.

The Grothendieck group can also be constructed using generators and relations: denoting by $(Z(M),+')$ the free abelian group generated by the set M, the Grothendieck group is the quotient of $Z(M)$ by the subgroup generated by $\{x +' y -' (x+y) \mid x, y \in M\}$.

Generalization

To apply the Grothendieck group to purely algebraic settings, it is useful to generalize it to the case of an essentially small abelian category. To do this, let $\mathcal{A}$ be an essentially small abelian category. Let F be the free abelian group generated by isomorphism classes of objects of the category. (This is where the hypothesis of essential smallness is necessary; without it, F would not be a set.) We will impose some relations on F. Call R the subgroup of F generated as follows: For each exact sequence $0 \to A \to B \to C \to 0$ in $\mathcal{A}$, the element

$[A] + [C] - [B]$

is in R. Then the Grothendieck group $K_0(\mathcal{A})$ is F/R.

K_0 of an abelian category has a similar universal property to K_0 of a commutative monoid. We make a preliminary definition: A function χ from isomorphism classes of objects of an abelian category $\mathcal{A}$ to an abelian group A is called *additive* if, for each exact sequence $0 \to A \to B \to C \to 0$, we have $\chi(A) + \chi(C) - \chi(B) = 0$. Then, for any additive function χ: $\mathcal{A} \to A$, there is a unique abelian group homomorphism f: $K_0\mathcal{A} \to A$ such that χ factors through f and the map that takes each object of $\mathcal{A}$ to the element representing its isomorphism class in $K_0(\mathcal{A})$.

This universal property makes $K_0(\mathcal{A})$ the 'universal receiver' of generalized Euler characteristics. In particular, for every bounded complex of objects in $\mathcal{A}$

$$\cdots \to 0 \to 0 \to A^n \to A^{n+1} \to \cdots \to A^{m-1} \to A^m \to 0 \to 0 \to \cdots$$

we have a canonical element

$$[A^*] = \sum_i (-1)^i [A^i] = \sum_i (-1)^i [H^i(A^*)] \in K_0.$$

In fact the Grothendieck group was originally introduced for the study of Euler characteristics.

Splitting principle

The relationship between K_0 of a commutative monoid and K_0 of an abelian category comes from the splitting principle. According to the splitting principle, we can always take an exact sequence $0 \to A \to B \to C \to 0$ and find a closely related exact sequence in which the middle term splits, that is, it is the direct sum of the other two terms. Because of this, the Grothendieck group of the commutative monoid of vector bundles on a smooth manifold is the same as the Grothendieck group of the abelian category of vector bundles on that same smooth manifold.

K_0 is often defined for a ring or for a ringed space. The usual construction is as follows: For a not necessarily commutative ring R, one lets the abelian category $\mathcal{A}$ be the category of all finitely generated projective modules over the ring. For a ringed space (X, O_X), one lets the abelian category $\mathcal{A}$ be the category of all coherent sheaves on X. This makes K_0 into a functor.

There is another Grothendieck group of a ring or a ringed space which is sometimes useful. The Grothendieck group G_0 of a ring is the Grothendieck group associated to the category of all finitely generated modules over a ring. Similarly, the Grothendieck group G_0 of a ringed space is the Grothendieck group associated to the category of all quasicoherent sheaves on the ringed space. G_0 is *not* a functor, but nevertheless it carries important information.

Example

The easiest example of the Grothendieck group construction is the construction of the integers from the natural numbers. Let us first check that the natural numbers, $\mathbb{N}$, indeed form a monoid.
This is easy, the operation should be regular addition and the identity element is zero. We know that addition is associative and indeed $0+n=n+0=n$ for any natural number.
Now when we use the Grothendieck group construction we obtain the formal differences between natural numbers as elements $n-m$ and we have the equivalence relation

$$n-m \sim n'-m' \leftrightarrow n+m'=n'+m \cdot$$

Now define

$$n := [n-0],$$
$$-n := [0-n]$$

for all $n \in \mathbb{N}$. This defines the integers $\mathbb{Z}$.

In the abelian category of finite dimensional vector spaces over a field k , two vector spaces are isomorphic if and only if they have the same dimension. Thus, for a vector space V the class $[V]=[k^{\dim(V)}]$ in $K_0(\mathrm{Vect}_{\mathrm{fin}})$.
Moreover for an exact sequence

$$0 \to k^l \to k^m \to k^n \to 0$$

$m = l + n$, so

$$[k^{l+n}]=[k^l]+[k^n]=(l+n)[k].$$

Thus $[V]=\dim(V)[k]$, the Grothendieck group $K_0(\mathrm{Vect}_{\mathrm{fin}})$ is isomorphic to $\mathbb{Z}$ and is generated by *[k]*.
Finally for a bounded complex of finite dimensional vector spaces V^* ,

$$[V^*]=\chi(V^*)[k]$$

where χ is the standard Euler characteristic defined by

$$\chi(V^*)=\sum_i(-1)^i\dim V=\sum_i(-1)^i\dim H^i(V^*).$$

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[1] http://planetmath.org/?op=getobj&from=objects&id=4290

Esquisse d'un Programme

"Esquisse d'un Programme" is a famous proposal for long-term mathematical research made by the German-born, French mathematician Alexander Grothendieck [1] . He pursued the sequence of logically linked ideas in his important project proposal from 1984 until 1988, but his proposed research continues to date to be of major interest in several branches of advanced mathematics. Grothendieck's vision provides inspiration today for several developments in mathematics such as the extension and generalization of Galois theory, which is currently being extended based on his original proposal.

Brief history

Submitted in 1984, the *Esquisse d'un Programme*[2] was a successful proposal submitted by Alexander Grothendieck for a position at the Centre National de la Recherche Scientifique, which Grothendieck held from 1984 till 1988.[3] This proposal was however not formally published until 1997, because the author "could not be found, much less his permission requested".[4] The outlines of *dessins d'enfants*, or "children's drawings", and "anabelian geometry", that are contained in this manuscript continue to inspire research.

Abstract of Grothendieck's programme

("*Sommaire*")

- 1. The Proposal and enterpise ("Envoi").
- 2. "Teichmüller's Lego-game and the Galois group of Q over Q" ("Un jeu de “Lego-Teichmüller” et le groupe de Galois de Q sur Q").
- 3. Number fields associated with "dessin d'enfants". ("Corps de nombres associés à un dessin d’enfant").
- 4. Regular polyhedra over finite fields ("Polyèdres réguliers sur les corps finis").
- 5. General topology or a 'Moderated Topology' ("Haro sur la topologie dite 'générale', et réflexions heuristiques vers une topologie dite 'modérée").
- 6. Differentiable theories and moderated theories ("Théories différentiables" (à la Nash) et “théories modérées").
- 7. Pursuing Stacks ("À la Poursuite des Champs"}[5] .
- 8. Two-dimensional geometry ("Digressions de géométrie bidimensionnelle"[6] .
- 9. Summary of proposed studies ("Bilan d’une activité enseignante").
- 10. Epilogue.
- Notes

Suggested further reading for the interested mathematical reader is provided in the *References* section.

Extensions of Galois's theory for groups: Galois groupoids, categories and functors

Galois has developed a powerful, fundamental algebraic theory in mathematics that provides very efficient computations for certain algebraic problems by utilizing the algebraic concept of groups, which is now known as the theory of Galois groups; such computations were not possible before, and also in many cases are much more effective than the 'direct' calculations without using groups[7] . To begin with, Alexander Grothendieck stated in his proposal: *"Thus, the group of Galois is realized as the automorphism group of a concrete, pro-finite group which respects certain structures that are essential to this group."* This fundamental, Galois group theory in mathematics has been considerably expanded, at first to groupoids- as proposed in Alexander Grothendieck's *Esquisse d' un Programme* (*EdP*)- and now already partially carried out for groupoids; the latter are now further developed beyond groupoids to categories by several groups of mathematicians. Here, we shall focus only on the well-established and fully validated extensions of Galois' theory. Thus, EdP also proposed and anticipated, along previous Alexander Grothendieck's *IHÉS* seminars (SGA1 to SGA4) held in the 1960s, the development of even more powerful extensions of the original Galois's theory for groups by utilizing categories, functors and natural transformations, as well as further expansion of the manifold of ideas presented in Alexander Grothendieck's *Descent Theory*. The notion of motive has also been pursued actively. This was developed into the motivic Galois group, Grothendieck topology and Grothendieck category [8] . Such developments were recently extended in algebraic topology *via* representable functors and the fundamental groupoid functor.

Notes

[1] Scharlau, Winifred (September 2008), written at Oberwolfach, Germany, "Who is Alexander Grothendieck", *Notices of the American Mathematical Society* (Providence, RI: American Mathematical Society) 55(8): 930–941, ISSN 1088-9477, OCLC 34550461, http://www.ams.org/notices/200808/tx080800930p.pdf

[2] Alexander Grothendieck, 1984. "Esquisse d'un Programme", (1984 manuscript), finally published in Schneps and Lochak (1997, I), pp.5-48; English transl., ibid., pp. 243-283. MR 99c:14034

[3] Rehmeyer, Julie (May 9, 2008), "Sensitivity to the Harmony of Things", *Science News*

[4] Schneps and Lochak (1997, I) p.1

[5] http://www.bangor.ac.uk/r.brown/pstacks.htm

[6] Cartier, Pierre (2001), "A mad day's work: from Grothendieck to Connes and Kontsevich The evolution of concepts of space and symmetry", *Bull. Amer. Math. Soc.* **38**(4): 389–408, <http://www.ams.org/bull/2001-38-04/S0273-0979-01-00913-2/S0273-0979-01-00913-2.pdf>. An English translation of Cartier (1998)

[7] Cartier, Pierre (1998), "La Folle Journée, de Grothendieck à Connes et Kontsevich — Évolution des Notions d'Espace et de Symétrie", *Les Relations entre les Mathématiques et la Physique Théorique — Festschrift for the 40th anniversary of the* IHÉS, *Institut des Hautes Études Scientifiques*, pp. 11–19

[8] http://planetmath.org/encyclopedia/GrothendieckCategory.html

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  - Jean-Pierre Serre. 1964. Cohomologie Galoisienne, Springer-Verlag: Berlin.
  - J. L. Verdier. 1965. Algèbre homologiques et Catégories derivées. North Holland Publ. Cie).
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External links

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- The best rejected proposal ever (http://www.neverendingbooks.org/index.php/the-best-rejected-proposal-ever.html), Never Ending Books, Lieven le Bruyn

Galois theory

In mathematics, more specifically in abstract algebra, **Galois theory**, named after Évariste Galois, provides a connection between field theory and group theory. Using Galois theory, certain problems in field theory can be reduced to group theory, which is in some sense simpler and better understood.


Évariste Galois (1811–1832)

Originally Galois used permutation groups to describe how the various roots of a given polynomial equation are related to each other. The modern approach to Galois theory, developed by Richard Dedekind, Leopold Kronecker and Emil Artin, among others, involves studying automorphisms of field extensions.

Further abstraction of Galois theory is achieved by the theory of Galois connections.

Application to classical problems

The birth of Galois theory was originally motivated by the following question, whose answer is known as the Abel–Ruffini theorem.

> *"Why is there no formula for the roots of a fifth (or higher) degree polynomial equation in terms of the coefficients of the polynomial, using only the usual algebraic operations (addition, subtraction, multiplication, division) and application of radicals (square roots, cube roots, etc)?"*

Galois theory not only provides a beautiful answer to this question, it also explains in detail why it *is* possible to solve equations of degree four or lower in the above manner, and why their solutions take the form that they do. Further, it gives a conceptually clear, and often practical, means of telling when some particular equation of higher degree can be solved in that manner.

Galois theory also gives a clear insight into questions concerning problems in compass and straightedge construction. It gives an elegant characterisation of the ratios of lengths that can be constructed with this method. Using this, it becomes relatively easy to answer such classical problems of geometry as

> *"Which regular polygons are constructible polygons?"*
>
> *"Why is it not possible to trisect every angle?"*

History

Galois theory originated in the study of symmetric functions – the coefficients of a polynomial are (up to sign) the elementary symmetric polynomials in the roots. For instance, $(x - a)(x - b) = x^2 - (a + b)x + ab$, where 1, $a + b$ and ab are the elementary polynomials of degree 0, 1 and 2 in two variables.

This was first formalized by the 16th century French mathematician François Viète, in Viète's formulas, for the case of positive real roots. In the opinion of the 18th century British mathematician Charles Hutton,[1] the expression of coefficients of a polynomial in terms of the roots (not only for positive roots) was first understood by the 17th century French mathematician Albert Girard; Hutton writes:

> ...[Girard was] the first person who understood the general doctrine of the formation of the coefficients of the powers from the sum of the roots and their products. He was the first who discovered the rules for summing the powers of the roots of any equation.

In this vein, the discriminant is a symmetric function in the roots which reflects properties of the roots – it is zero if and only if the polynomial has a multiple root, and for quadratic and cubic polynomials it is positive if and only if all roots are real and distinct, and negative if and only if there is a pair of distinct complex conjugate roots. See Discriminant: nature of the roots for details.

The cubic was first partly solved by the 15th/16th century Italian mathematician Scipione del Ferro, who did not however publish his results; this method only solved one of three classes, as the others involved taking square roots of negative numbers, and complex numbers were not known at the time. This solution was then rediscovered independently in 1535 by Niccolò Fontana Tartaglia, who shared it with Gerolamo Cardano, asking him to not publish it. Cardano then extended this to the other two cases, using square roots of negatives as intermediate steps; see details at Cardano's method. After the discovery of Ferro's work, he felt that Tartaglia's method was no longer secret, and thus he published his complete solution in his 1545 *Ars Magna.* His student Lodovico Ferrari solved the quartic polynomial, which solution Cardano also included in *Ars Magna.*

A further step was the 1770 paper *Réflexions sur la résolution algébrique des équations* by the French-Italian mathematician Joseph Louis Lagrange, in his method of Lagrange resolvents, where he analyzed Cardano and Ferrarri's solution of cubics and quartics by considering them in terms of *permutations* of the roots, which yielded an auxiliary polynomial of lower degree, providing a unified understanding of the solutions and laying the groundwork for group theory and Galois theory. Crucially, however, he did not consider *composition* of permutations. Lagrange's method did not extend to quintic equations or higher, because the resolvent had higher degree.

The quintic was almost proven to have no general solutions by radicals by Paolo Ruffini in 1799, whose key insight was to use permutation *groups*, not just a single permutation. His solution contained a gap, which Cauchy considered minor, though this was not patched until the work of Norwegian mathematician Niels Henrik Abel, who published a proof in 1824, thus establishing the Abel–Ruffini theorem.

While Ruffini and Abel established that the *general* quintic could not be solved, some *particular* quintics can be solved, such as $(x - 1)^5$, and the precise criterion by which a *given* quintic or higher polynomial could be determined to be solvable or not was given by Évariste Galois, who showed that whether a polynomial was solvable or not was equivalent to whether or not the permutation group of its roots – in modern terms, its Galois group – had a certain structure – in modern terms, whether or not it was a solvable group. This group was always solvable for polynomials of degree four or less, but not always so for polynomials of degree five and greater, which explains why there is no general solution in higher degree.

The permutation group approach to Galois theory

Given a polynomial, it may be that some of the roots are connected by various algebraic equations. For example, it may be that for two of the roots, say A and B, that $A^2 + 5B^3 = 7$. The central idea of Galois theory is to consider those permutations (or rearrangements) of the roots having the property that *any* algebraic equation satisfied by the roots is *still satisfied* after the roots have been permuted. An important proviso is that we restrict ourselves to algebraic equations whose coefficients are rational numbers. (One might instead specify a certain field in which the coefficients should lie but, for the simple examples below, we will restrict ourselves to the field of rational numbers.)

These permutations together form a permutation group, also called the Galois group of the polynomial (over the rational numbers). To illustrate this point, consider the following examples:

First example — a quadratic equation

Consider the quadratic equation

$$x^2 - 4x + 1 = 0.$$

By using the quadratic formula, we find that the two roots are

$$A = 2 + \sqrt{3}$$
$$B = 2 - \sqrt{3}.$$

Examples of algebraic equations satisfied by A and B include

$$A + B = 4,$$

and

$$AB = 1.$$

Obviously, in either of these equations, if we exchange A and B, we obtain another true statement. For example, the equation $A + B = 4$ becomes simply $B + A = 4$. Furthermore, it is true, but far less obvious, that this holds for *every* possible algebraic equation with rational coefficients satisfied by the roots A and B; to prove this requires the theory of symmetric polynomials.

We conclude that the Galois group of the polynomial $x^2 - 4x + 1$ consists of two permutations: the identity permutation which leaves A and B untouched, and the transposition permutation which exchanges A and B. It is a cyclic group of order two, and therefore isomorphic to $\mathbf{Z}/2\mathbf{Z}$.

One might object that A and B are related by yet another algebraic equation,

$$A - B - 2\sqrt{3} = 0$$

which does *not* remain true when A and B are exchanged. However, this equation does not concern us, because it does not have rational coefficients; in particular, $-2\sqrt{3}$ is not rational.

A similar discussion applies to any quadratic polynomial $ax^2 + bx + c$, where a, b and c are rational numbers.

- If the polynomial has only one root, for example $x^2 - 4x + 4 = (x-2)^2$, then the Galois group is trivial; that is, it contains only the identity permutation.
- If it has two distinct *rational* roots, for example $x^2 - 3x + 2 = (x-2)(x-1)$, the Galois group is again trivial.
- If it has two *irrational* roots (including the case where the roots are complex), then the Galois group contains two permutations, just as in the above example.

Second example

Consider the polynomial

$$x^4 - 10x^2 + 1,$$

which can also be written as

$$(x^2 - 5)^2 - 24.$$

We wish to describe the Galois group of this polynomial, again over the field of rational numbers. The polynomial has four roots:

$$A = \sqrt{2} + \sqrt{3}$$
$$B = \sqrt{2} - \sqrt{3}$$
$$C = -\sqrt{2} + \sqrt{3}$$
$$D = -\sqrt{2} - \sqrt{3}.$$

There are 24 possible ways to permute these four roots, but not all of these permutations are members of the Galois group. The members of the Galois group must preserve any algebraic equation with rational coefficients involving A, B, C and D. One such equation is

$$A + D = 0.$$

However, since

$$A + C = 2\sqrt{3} \neq 0,$$

the permutation

$$(A, B, C, D) \to (A, B, D, C)$$

is not permitted (because it transforms the valid equation $A + D = 0$ into the invalid equation $A + C = 0$).

Another equation that the roots satisfy is

$$(A + B)^2 = 8.$$

This will exclude further permutations, such as

$$(A, B, C, D) \to (A, C, B, D).$$

Continuing in this way, we find that the only permutations (satisfying both equations simultaneously) remaining are

$$(A, B, C, D) \to (A, B, C, D)$$
$$(A, B, C, D) \to (C, D, A, B)$$
$$(A, B, C, D) \to (B, A, D, C)$$
$$(A, B, C, D) \to (D, C, B, A),$$

and the Galois group is isomorphic to the Klein four-group.

The modern approach by field theory

In the modern approach, one starts with a field extension L/K (read: L over K), and examines the group of field automorphisms of L/K (these are mappings α: $L \to L$ with $\alpha(x) = x$ for all x in K). See the article on Galois groups for further explanation and examples.

The connection between the two approaches is as follows. The coefficients of the polynomial in question should be chosen from the base field K. The top field L should be the field obtained by adjoining the roots of the polynomial in question to the base field. Any permutation of the roots which respects algebraic equations as described above gives rise to an automorphism of L/K, and vice versa.

In the first example above, we were studying the extension $\mathbf{Q}(\sqrt{3})/\mathbf{Q}$, where $\mathbf{Q}$ is the field of rational numbers, and $\mathbf{Q}(\sqrt{3})$ is the field obtained from $\mathbf{Q}$ by adjoining $\sqrt{3}$. In the second example, we were studying the extension

Q(A,B,C,D)/**Q**.

There are several advantages to the modern approach over the permutation group approach.

- It permits a far simpler statement of the fundamental theorem of Galois theory.
- The use of base fields other than **Q** is crucial in many areas of mathematics. For example, in algebraic number theory, one often does Galois theory using number fields, finite fields or local fields as the base field.
- It allows one to more easily study infinite extensions. Again this is important in algebraic number theory, where for example one often discusses the absolute Galois group of **Q**, defined to be the Galois group of K/**Q** where K is an algebraic closure of **Q**.
- It allows for consideration of inseparable extensions. This issue does not arise in the classical framework, since it was always implicitly assumed that arithmetic took place in characteristic zero, but nonzero characteristic arises frequently in number theory and in algebraic geometry.
- It removes the rather artificial reliance on chasing roots of polynomials. That is, different polynomials may yield the same extension fields, and the modern approach recognizes the connection between these polynomials.

Solvable groups and solution by radicals

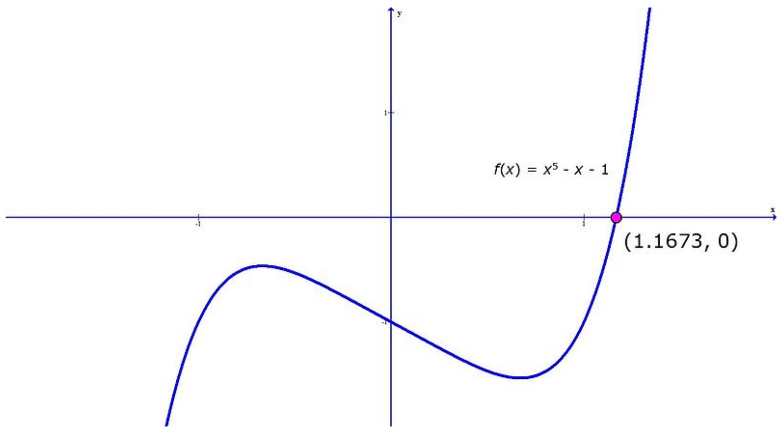
The notion of a solvable group in group theory allows one to determine whether a polynomial is solvable in the radicals, depending on whether its Galois group has the property of solvability. In essence, each field extension L/K corresponds to a factor group in a composition series of the Galois group. If a factor group in the composition series is cyclic of order n, then if the corresponding field extension is an extension of a field containing a primitive root of unity, then it is a radical extension, and the elements of L can then be expressed using the nth root of some element of K.

If all the factor groups in its composition series are cyclic, the Galois group is called *solvable*, and all of the elements of the corresponding field can be found by repeatedly taking roots, products, and sums of elements from the base field (usually **Q**).

One of the great triumphs of Galois Theory was the proof that for every $n > 4$, there exist polynomials of degree n which are not solvable by radicals—the Abel–Ruffini theorem. This is due to the fact that for $n > 4$ the symmetric group S_n contains a simple, non-cyclic, normal subgroup.

A non-solvable quintic example

Van der Waerden cites the polynomial $f(x) = x^5 - x - 1$. By the rational root theorem it has no rational zeros. Neither does it have linear factors modulo 2 or 3.



The polynomial $f(x) = x^5 - x - 1$. The lone real root x =1.1673... is algebraic, but not expressible as radicals. The other four roots are complex numbers.

$f(x)$ has the factorization $(x^2+x+1)(x^3+x^2+1)$ modulo 2. That means its Galois group modulo 2 is cyclic of order 6.

$f(x)$ has no quadratic factor modulo 3. Thus its Galois group modulo 3 has order 5.

We know[2] that a Galois group modulo a prime is isomorphic to a subgroup of the Galois group over the rationals. A permutation group on 5 objects with operations of orders 6 and 5 must be the symmetric group S_5, which must be the Galois group of $f(x)$. This is one of the simplest examples of a non-solvable quintic polynomial. Serge Lang said that Artin was fond of this example.

The inverse Galois problem

All finite groups do occur as Galois groups. It is easy to construct field extensions with any given finite group as Galois group, as long as one does not also specify the ground field.

For that, choose a field K and a finite group G. Cayley's theorem says that G is (up to isomorphism) a subgroup of the symmetric group S on the elements of G. Choose indeterminates $\{x_\alpha\}$, one for each element α of G, and adjoin them to K to get the field $F = K(\{x_\alpha\})$. Contained within F is the field L of symmetric rational functions in the $\{x_\alpha\}$. The Galois group of F/L is S, by a basic result of Emil Artin. G acts on F by restriction of action of S. If the fixed field of this action is M, then, by the fundamental theorem of Galois theory, the Galois group of F/M is G.

It is an open problem to prove the existence of a field extension of the rational field **Q** with a given finite group as Galois group. Hilbert played a part in solving the problem for all symmetric and alternating groups. Igor Shafarevich proved that every solvable finite group is the Galois group of some extension of **Q**. Various people have solved the inverse Galois problem for selected non-abelian simple groups. Existence of solutions has been shown for all but possibly one (Mathieu group M_{23}) of the 26 sporadic simple groups. There is even a polynomial with integral coefficients whose Galois group is the Monster group.

Notes

[1] (Funkhouser 1930)
[2] V.V. Praslov, Polynomials. (2004), Theorem 5.4.5(a)

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External links

Some on-line tutorials on Galois theory appear at:

- http://www.math.niu.edu/~beachy/aaol/galois.html
- http://nrich.maths.org/public/viewer.php?obj_id=1422
- http://www.jmilne.org/math/CourseNotes/ft.html

Online textbooks in French, German, Italian and English can be found at:

- http://www.galois-group.net/

Grothendieck's Galois theory

In mathematics, **Grothendieck's Galois theory** is a highly abstract approach to the Galois theory of fields, developed around 1960 to provide a way to study the fundamental group of algebraic topology in the setting of algebraic geometry. It provides, in the classical setting of field theory, an alternative perspective to that of Emil Artin based on linear algebra, which became standard from about the 1930s.

The approach of Alexander Grothendieck is concerned with the category-theoretic properties that characterise the categories of finite *G*-sets for a fixed profinite group *G*. For example, *G* might be the group denoted $\hat{\mathbb{Z}}$, which is the inverse limit of the cyclic additive groups **Z**/n**Z** — or equivalently the completion of the infinite cyclic group **Z** for the topology of subgroups of finite index. A finite *G*-set is then a finite set *X* on which *G* acts through a quotient finite cyclic group, so that it is specified by giving some permutation of *X*.

In the above example, a connection with classical Galois theory can be seen by regarding $\hat{\mathbb{Z}}$ as the profinite Galois group Gal(F/F) of the algebraic closure F of any finite field *F*, over *F*. That is, the automorphisms of F fixing *F* are described by the inverse limit, as we take larger and larger finite splitting fields over *F*. The connection with geometry can be seen when we look at covering spaces of the unit disk in the complex plane with the origin removed: the finite covering realised by the z^n map of the disk, thought of by means of a complex number variable *z*, corresponds to the subgroup *n*.**Z** of the fundamental group of the punctured disk.

The theory of Grothendieck, published in SGA1, shows how to reconstruct the category of *G*-sets from a *fibre functor* Φ, which in the geometric setting takes the fibre of a covering above a fixed base point (as a set). In fact there is an isomorphism proved of the type

$$G \cong \mathrm{Aut}(\Phi),$$

the latter being the group of automorphisms (self-natural equivalences) of Φ. An abstract classification of categories with a functor to the category of sets is given, by means of which one can recognise categories of *G*-sets for *G* profinite.

To see how this applies to the case of fields, one has to study the tensor product of fields. Later developments in topos theory make this all part of a theory of *atomic toposes*.

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Notes on Grothendieck's Galois Theory http://arxiv.org/abs/math/0009145v1

Galois cohomology

In mathematics, **Galois cohomology** is the study of the group cohomology of Galois modules, that is, the application of homological algebra to modules for Galois groups. A Galois group *G* associated to a field extension *L*/*K* acts in a natural way on some abelian groups, for example those constructed directly from *L*, but also through other Galois representations that may be derived by more abstract means. Galois cohomology accounts for the way in which taking Galois-invariant elements fails to be an exact functor.

The current theory of Galois cohomology came together around 1950, when it was realised that the Galois cohomology of idele class groups in algebraic number theory was one way to formulate class field theory, at the time in the process of ridding itself of connections to L-functions. Galois cohomology makes no assumption that Galois groups are abelian groups, so that this was a non-abelian theory. It was formulated abstractly as a theory of class formations. Two developments of the 1960s turned the position around. Firstly, Galois cohomology appeared as the foundational layer of étale cohomology theory (roughly speaking, the theory as it applies to zero-dimensional schemes). Secondly, non-abelian class field theory was launched as part of the Langlands philosophy, which meant that L-functions were back, with a vengeance.

The earliest results identifiable as Galois cohomology had been known long before, in algebraic number theory and the arithmetic of elliptic curves. The normal basis theorem implies that the first cohomology group of the additive group of *L* will vanish; this is a result on general field extensions, but was known in some form to Richard Dedekind. The corresponding result for the multiplicative group is known as Hilbert's Theorem 90, and was known before 1900. Kummer theory was another such early part of the theory, giving a description of the connecting homomorphism coming from the *m*-th power map.

In fact for a while the multiplicative case of a 1-cocycle for groups that are not necessarily cyclic was formulated as the solubility of **Noether's equations**, named for Emmy Noether; they appear under this name in Emil Artin's treatment of Galois theory, and may have been folklore in the 1920s. The case of 2-cocycles for the multiplicative group is that of the Brauer group, and the implications seem to have been well known to algebraists of the 1930s.

In another direction, that of torsors, these were already implicit in the infinite descent arguments of Fermat for elliptic curves. Numerous direct calculations were done, and the proof of the Mordell–Weil theorem had to proceed by some surrogate of a finiteness proof for a particular H^1 group. The 'twisted' nature of objects over fields that are not algebraically closed, which are not isomorphic but become so over the algebraic closure, was also known in many cases linked to other algebraic groups (such as quadratic forms, simple algebras, Severi–Brauer varieties), in the 1930s, before the general theory arrived.

The needs of number theory were in particular expressed by the requirement to have control of a local-global principle for Galois cohomology. This was formulated by means of results in class field theory, such as Hasse's norm theorem. In the case of elliptic curves it led to the key definition of the Tate–Shafarevich group in the Selmer group, which is the obstruction to the success of a local-global principle. Despite its great importance, for example in the Birch and Swinnerton-Dyer conjecture, it proved very difficult to get any control of it, until results of Karl Rubin gave a way to show in some cases it was finite (a result generally believed, since its conjectural order was predicted by an L-function formula).

The other major development of the theory, also involving John Tate was the Tate–Poitou duality result.

Technically speaking, G may be a profinite group, in which case the definitions need to be adjusted to allow only continuous cochains.

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Homological algebra

Homological algebra is the branch of mathematics which studies homology in a general algebraic setting. It is a relatively young discipline, whose origins can be traced to investigations in combinatorial topology (a precursor to algebraic topology) and abstract algebra (theory of modules and syzygies) at the end of the 19th century, chiefly by Henri Poincaré and David Hilbert.

The development of homological algebra was closely intertwined with the emergence of category theory. By and large, homological algebra is the study of homological functors and the intricate algebraic structures that they entail. One quite useful and ubiquitous concept in mathematics is that of **chain complexes**, which can be studied both through their homology and cohomology. Homological algebra affords the means to extract information contained in these complexes and present it in the form of homological invariants of rings, modules, topological spaces, and other 'tangible' mathematical objects. A powerful tool for doing this is provided by spectral sequences.

From its very origins, homological algebra has played an enormous role in algebraic topology. Its sphere of influence has gradually expanded and presently includes commutative algebra, algebraic geometry, algebraic number theory, representation theory, mathematical physics, operator algebras, complex analysis, and the theory of partial differential equations. K-theory is an independent discipline which draws upon methods of homological algebra, as does the noncommutative geometry of Alain Connes.

Chain complexes and homology

The **chain complex** is the central notion of homological algebra. It is a sequence $(C_\bullet, d_\bullet)$ of abelian groups and group homomorphisms, with the property that the composition of any two consecutive maps is zero:

$$C_\bullet : \cdots \longrightarrow C_{n+1} \xrightarrow{d_{n+1}} C_n \xrightarrow{d_n} C_{n-1} \xrightarrow{d_{n-1}} \cdots, \quad d_n \circ d_{n+1} = 0.$$

The elements of C_n are called n-**chains** and the homomorphisms d_n are called the **boundary maps** or **differentials**. The **chain groups** C_n may be endowed with extra structure; for example, they may be vector spaces or modules over a fixed ring *R*. The differentials must preserve the extra structure if it exists; for example, they must be linear maps or homomorphisms of *R*-modules. For notational convenience, restrict attention to abelian groups (more correctly, to the category **Ab** of abelian groups); a celebrated theorem by Barry Mitchell implies the results will generalize to any abelian category. Every chain complex defines two further sequences of abelian groups, the **cycles** $Z_n = \mathrm{Ker}\ d_n$ and the **boundaries** $B_n = \mathrm{Im}\ d_{n+1}$, where Ker *d* and Im *d* denote the kernel and the image of *d*. Since the composition of two consecutive boundary maps is zero, these groups are embedded into each other as

$$B_n \subseteq Z_n \subseteq C_n.$$

Subgroups of abelian groups are automatically normal; therefore we can define the *n*th **homology group** $H_n(C)$ as the factor group of the *n*-cycles by the *n*-boundaries,

$$H_n(C) = Z_n/B_n = \mathrm{Ker}\, d_n/\mathrm{Im}\, d_{n+1}.$$

A chain complex is called **acyclic** or an **exact sequence** if all its homology groups are zero.

Chain complexes arise in abundance in algebra and algebraic topology. For example, if *X* is a topological space then the singular chains $C_n(X)$ are formal linear combinations of continuous maps from the standard *n*-simplex into *X*; if *K* is a simplicial complex then the simplicial chains $C_n(K)$ are formal linear combinations of the *n*-simplices of *X*; if $A = F/R$ is a presentation of an abelian group *A* by generators and relations, where *F* is a free abelian group spanned by the generators and *R* is the subgroup of relations, then letting $C_1(A) = R$, $C_0(A) = F$, and $C_n(A) = 0$ for all other *n* defines a sequence of abelian groups. In all these cases, there are natural differentials d_n making C_n into a chain complex, whose homology reflects the structure of the topological space *X*, the simplicial complex *K*, or the abelian group *A*. In the case of topological spaces, we arrive at the notion of singular homology, which plays a fundamental role in investigating the properties of such spaces, for example, manifolds.

On a philosophical level, homological algebra teaches us that certain chain complexes associated with algebraic or geometric objects (topological spaces, simplicial complexes, *R*-modules) contain a lot of valuable algebraic information about them, with the homology being only the most readily available part. On a technical level, homological algebra provides the tools for manipulating complexes and extracting this information. Here are two general illustrations.

- Two objects *X* and *Y* are connected by a map *f* between them. Homological algebra studies the relation, induced by the map *f*, between chain complexes associated with *X* and *Y* and their homology. This is generalized to the case of several objects and maps connecting them. Phrased in the language of category theory, homological algebra studies the functorial properties of various constructions of chain complexes and of the homology of these complexes.
- An object *X* admits multiple descriptions (for example, as a topological space and as a simplicial complex) or the complex $C_\bullet(X)$ is constructed using some 'presentation' of *X*, which involves non-canonical choices. It is important to know the effect of change in the description of *X* on chain complexes associated with *X*. Typically, the complex and its homology $H_\bullet(C)$ are functorial with respect to the presentation; and the homology (although not the complex itself) is actually independent of the presentation chosen, thus it is an invariant of *X*.

Functoriality

A continuous map of topological spaces gives rise to a homomorphism between their *n*th homology groups for all *n*. This basic fact of algebraic topology finds a natural explanation through certain properties of chain complexes. Since it is very common to study several topological spaces simultaneously, in homological algebra one is led to simultaneous consideration of multiple chain complexes.

A **morphism** between two chain complexes, $F : C_\bullet \to D_\bullet$, is a family of homomorphisms of abelian groups $F_n : C_n \to D_n$ that commute with the differentials, in the sense that $F_{n-1} \circ d_n^C = d_n^D \circ F_n$ for all n. A morphism of chain complexes induces a morphism $H_\bullet(F)$ of their homology groups, consisting of the homomorphisms $H_n(F) : H_n(C) \to H_n(D)$ for all n. A morphism F is called a **quasi-isomorphism** if it induces an isomorphism on the nth homology for all n.

Many constructions of chain complexes arising in algebra and geometry, including singular homology, have the following functoriality property: if two objects X and Y are connected by a map f, then the associated chain complexes are connected by a morphism $F = C(f)$ from $C_\bullet(X)$ to $C_\bullet(Y)$, and moreover, the composition $g \circ f$ of maps $f: X \to Y$ and $g: Y \to Z$ induces the morphism $C(g \circ f)$ from $C_\bullet(X)$ to $C_\bullet(Z)$ that coincides with the composition $C(g) \circ C(f)$. It follows that the homology groups $H_\bullet(C)$ are functorial as well, so that morphisms between algebraic or topological objects give rise to compatible maps between their homology.

The following definition arises from a typical situation in algebra and topology. A triple consisting of three chain complexes $L_\bullet, M_\bullet, N_\bullet$ and two morphisms between them, $f : L_\bullet \to M_\bullet, g : M_\bullet \to N_\bullet$, is called an **exact triple**, or a **short exact sequence of complexes**, and written as

$$0 \longrightarrow L_\bullet \xrightarrow{f} M_\bullet \xrightarrow{g} N_\bullet \longrightarrow 0,$$

if for any n, the sequence

$$0 \longrightarrow L_n \xrightarrow{f_n} M_n \xrightarrow{g_n} N_n \longrightarrow 0$$

is a short exact sequence of abelian groups. By definition, this means that f_n is an injection, g_n is a surjection, and Im f_n = Ker g_n. One of the most basic theorems of homological algebra, sometimes known as the zig-zag lemma, states that, in this case, there is a **long exact sequence in homology**

$$\dots \longrightarrow H_n(L) \xrightarrow{H_n(f)} H_n(M) \xrightarrow{H_n(g)} H_n(N) \xrightarrow{\delta_n} H_{n-1}(L) \xrightarrow{H_{n-1}(f)} H_{n-1}(M) \longrightarrow \dots,$$

where the homology groups of L, M, and N cyclically follow each other, and δ_n are certain homomorphisms determined by f and g, called the **connecting homomorphisms**. Topological manifestations of this theorem include the Mayer–Vietoris sequence and the long exact sequence for relative homology.

Foundational aspects

Cohomology theories have been defined for many different objects such as topological spaces, sheaves, groups, rings, Lie algebras, and C*-algebras. The study of modern algebraic geometry would be almost unthinkable without sheaf cohomology.

Central to homological algebra is the notion of exact sequence; these can be used to perform actual calculations. A classical tool of homological algebra is that of derived functor; the most basic examples are functors Ext and Tor.

With a diverse set of applications in mind, it was natural to try to put the whole subject on a uniform basis. There were several attempts before the subject settled down. An approximate history can be stated as follows:

- Cartan-Eilenberg: In their 1956 book "Homological Algebra", these authors used projective and injective module resolutions.
- 'Tohoku': The approach in a celebrated paper by Alexander Grothendieck which appeared in the Second Series of the Tohoku Mathematical Journal in 1957, using the abelian category concept (to include sheaves of abelian groups).

- The derived category of Grothendieck and Verdier. Derived categories date back to Verdier's 1967 thesis. They are examples of triangulated categories used in a number of modern theories.

These move from computability to generality.

The computational sledgehammer *par excellence* is the spectral sequence; these are essential in the Cartan-Eilenberg and Tohoku approaches where they are needed, for instance, to compute the derived functors of a composition of two functors. Spectral sequences are less essential in the derived category approach, but still play a role whenever concrete computations are necessary.

There have been attempts at 'non-commutative' theories which extend first cohomology as *torsors* (important in Galois cohomology).

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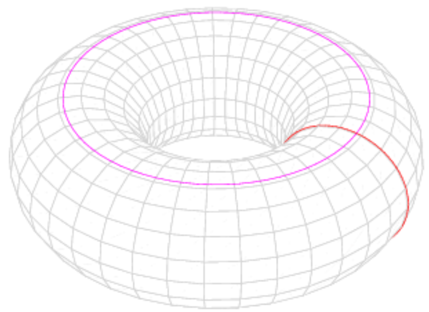
Homology theory

In mathematics, **homology theory** is the axiomatic study of the intuitive geometric idea of *homology of cycles* on topological spaces. It can be broadly defined as the study of homology theories on topological spaces.

The general idea



A torus with generators colored in pink and red.

To any topological space X and any natural number k, one can associate a set $H_k(X)$, whose elements are called (k-dimensional) homology classes. There is a well-defined way to add and subtract homology classes, which makes $H_k(X)$ into an abelian group, called the k th homology group of X. In heuristic terms, the size and structure of $H_k(X)$ gives information about the number of k-dimensional holes in X. For example, if X is a figure eight, then it has two holes, which in this context count as being one-dimensional. The corresponding homology group $H_1(X)$ can be identified with the group $\mathbb{Z} \oplus \mathbb{Z}$ of pairs of integers, with one copy of $\mathbb{Z}$ for each hole. While it seems very straightforward to say that X has two holes, it is surprisingly hard to formulate this in a mathematically rigorous way; this is a central purpose of homology theory.

For a more intricate example, if Y is a Klein bottle then $H_1(Y)$ can be identified with $\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$. This is not just a sum of copies of $\mathbb{Z}$, so it gives more subtle information than just a count of holes.

The formal definition of $H_1(X)$ can be sketched as follows. The elements of $H_1(X)$ are one-dimensional cycles, except that two cycles are considered to represent the same element if they are homologous. The simplest kind of one-dimensional cycles are just closed curves in X, which could consist of one or more loops. If a closed curve C_0 can be deformed continuously within X to another closed curve C_1, then C_0 and C_1 are homologous and so determine the same element of $H_1(X)$. This captures the main geometric idea, but the full definition is somewhat more complex. For details, see singular homology. There is also a version (called simplicial homology) that works when X is presented as a simplicial complex; this is smaller and easier to understand, but technically less flexible.

For example, let T be a torus, as shown on the right. Let C be the pink curve, and let D be the red one. For integers n and m, we have another closed curve that goes n times around C and then m times around D; this is denoted by $nC + mD$. It can be shown that any closed curve in T is homologous to $nC + mD$ for some n and m, and thus that $H_1(T)$ is again isomorphic to $\mathbb{Z} \oplus \mathbb{Z}$.

Cohomology

As well as the homology groups $H_k(X)$, one can define cohomology groups $H^k(X)$. In the common case where each group $H_k(X)$ is isomorphic to $\mathbb{Z}^{r_k}$ for some $r_k \in \mathbb{N}$, we just have $H^k(X) = Hom(H_k(X), \mathbb{Z})$, which is again isomorphic to $\mathbb{Z}^{r_k}$, and $H_k(X) = Hom(H^k(X), \mathbb{Z})$, so $H_k(X)$ and $H^k(X)$ determine each other. In general, the relationship between $H_k(X)$ and $H^k(X)$ is only a little more complicated, and is controlled by the universal coefficient theorem. The main advantage of cohomology over homology is that it has a natural ring structure: there is a way to multiply an i -dimensional cohomology class by a j -dimensional cohomology class to get an $i+j$ -dimensional cohomology class.

Applications

Notable theorems proved using homology include the following:

- The Brouwer fixed point theorem: If f is any continuous map from the ball B^n to itself, then there is a fixed point $a \in B^n$ with $f(a)=a$.
- Invariance of domain: If U is an open subset of $\mathbb{R}^n$ and $f : U \to \mathbb{R}^n$ is an injective continuous map, then $V = f(U)$ is open and f is a homeomorphism between U and V.
- The Hairy ball theorem: any vector field on the 2-sphere (or more generally, the $2k$ -sphere for any $k \geq 1$) vanishes at some point.
- The Borsuk–Ulam theorem: any continuous function from an *n*-sphere into Euclidean *n*-space maps some pair of antipodal points to the same point. (Two points on a sphere are called antipodal if they are in exactly opposite directions from the sphere's center.)

Intersection theory and Poincaré duality

Let M be a compact oriented manifold of dimension n . The Poincaré duality theorem gives a natural isomorphism $H^k(M) \simeq H_{n-k}(M)$, which we can use to transfer the ring structure from cohomology to homology. For any compact oriented submanifold $N \subseteq M$ of dimension d, one can define a so-called fundamental class $[N] \in H_d(M) \simeq H^{n-d}(M)$. If L is another compact oriented submanifold which meets N transversely, it works out that $[L][N] = [L \cap N]$. In many cases the group $H_d(M)$ will have a basis consisting of fundamental classes of submanifolds, in which case the product rule $[L][N] = [L \cap N]$ gives a very clear geometric picture of the ring structure.

Connection with integration

Suppose that X is an open subset of the complex plane, that $f(z)$ is a holomorphic function on X , and that C is a closed curve in X . There is then a standard way to define the contour integral $\oint_C f(z)dz$, which is a central idea in complex analysis. One formulation of Cauchy's integral theorem is as follows: if C_0 and C_1 are homologous, then $\oint_{C_0} f(z)dz = \oint_{C_1} f(z)dz$. (Many authors consider only the case where X is simply connected, in which case every closed curve is homologous to the empty curve and so $\oint_C f(z)dz = 0$.) This means that we can make sense of $\oint_c f(z)dz$ when c is merely a homology class, or in other words an element of $H_1(X)$. It is also important that in the case where $f(z)$ is the derivative of another function $g(z)$, we always have $\oint_C g'(z)dz = 0$ (even when C is not homologous to zero).

This is the simplest case of a much more general relationship between homology and integration, which is most efficiently formulated in terms of differential forms and de Rham cohomology. To explain this briefly, suppose that X is an open subset of $\mathbb{R}^N$, or more generally, that X is a manifold. One can then define objects called n -forms on X . If X is open in $\mathbb{R}^3$, then the 0-forms are just the scalar fields, the 1-forms are the vector fields, the 2-forms are the same as the 1-forms, and the 3-forms are the same as the 0-forms. There is also a kind of differentiation operation called the exterior derivative: if α is an n -form, then the exterior derivative is an $(n+1)$-form denoted by $d\alpha$. The standard operators div, grad and curl from vector calculus can be seen as special cases of this. There is a procedure for integrating an n -form α over an n -cycle C to get a number $\oint_C \alpha$

. It can be shown that $\oint_C d\beta = 0$ for any $(n-1)$-form β, and that $\oint_C \alpha$ depends only on the homology class of C, provided Stokes's Theorem and Divergence Theorem can be seen as special cases of this.

We say that α is *closed* if $d\alpha = 0$, and *exact* if $\alpha = d\beta$ for some β. It can be shown that $dd\beta$ is always zero, so that exact forms are always closed. The de Rham cohomology group $H^k_{dR}(X)$ is the quotient of the group of closed forms by the subgroup of exact forms. It follows from the above that there is a well-defined pairing $H_k(X) \times H^k_{dR}(X) \to \mathbb{R}$ given by integration.

Axiomatics and generalised homology

There are various different ways to define cohomology groups (for example singular cohomology, Čech cohomology, Alexander–Spanier cohomology or Sheaf cohomology). These give different answers for some exotic spaces, but there is a large class of spaces on which they all agree. This is most easily understood axiomatically: there is a list of properties known as the Eilenberg–Steenrod axioms, and any two constructions that share those properties will agree at least on all finite CW complexes, for example.

One of the axioms is the so-called dimension axiom: if P is a single point, then $H_n(P) = 0$ for all $n \neq 0$, and $H_0(P) = \mathbb{Z}$. We can generalise slightly by allowing an arbitrary abelian group A in dimension zero, but still insisting that the groups in nonzero dimension are trivial. It turns out that there is again an essentially unique system of groups satisfying these axioms, which are denoted by $H_*(X;A)$. In the common case where each group $H_k(X)$ is isomorphic to $\mathbb{Z}^{r_k}$ for some $r_k \in \mathbb{N}$, we just have $H_k(X;A) = A^{r_k}$. In general, the relationship between $H_k(X)$ and $H_k(X;A)$ is only a little more complicated, and is again controlled by the Universal coefficient theorem.

More significantly, we can drop the dimension axiom altogether. There are a number of different ways to define groups satisfying all the other axioms, including the following:

- The stable homotopy groups $\pi^S_k(X)$
- Various different flavours of cobordism groups: $MO_*(X)$, $MSO_*(X)$, $MU_*(X)$ and so on. The last of these (known as complex cobordism) is especially important, because of the link with formal group theory via a theorem of Daniel Quillen.
- Various different flavours of K-theory: $KO_*(X)$ (real periodic K-theory), $kO_*(X)$ (real connective), $KU_*(X)$ (complex periodic), $kU_*(X)$ (complex connective) and so on.
- Brown–Peterson homology, Morava K-theory, Morava E-theory, and other theories defined using the algebra of formal groups.
- Various flavours of elliptic homology

These are called generalised homology theories; they carry much richer information than ordinary homology, but are often harder to compute. Their study is tightly linked (via the Brown representability theorem) to stable homotopy.

Homological algebra and homology of other objects

A chain complex consists of groups C_i (for all $i \in \mathbb{Z}$) and homomorphisms $d : C_i \to C_{i-1}$ satisfying $dd = 0$. This condition shows that the groups $B_i = \mathrm{image}(d : C_{i+1} \to C_i)$ are contained in the groups $Z_i = \ker(d : C_i \to C_{i-1})$, so one can form the quotient groups $H_i = Z_i/B_i$, which are called the homology groups of the original complex. There is a similar theory of cochain complexes, consisting of groups C^i and homomorphisms $\delta : C^i \to C^{i+1}$. The simplicial, singular, Čech and Alexander–Spanier groups are all defined by first constructing a chain complex or cochain complex, an then taking its homology. Thus, a substantial part of the work in setting up these groups involves the general theory of chain and cochain complexes, which is known as homological algebra.

One can also associate (co)chain complexes to a wide variety of other mathematical objects, and then take their (co)homology. For example, there are cohomology modules for groups, Lie algebras and so on.

Notes

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Homotopical algebra

In mathematics, **homotopical algebra** is a collection of concepts comprising the *nonabelian* aspects of homological algebra as well as possibly the abelian aspects as special cases. The *homotopical* nomenclature stems from the fact that a common approach to such generalizations is via abstract homotopy theory and in particular the theory of closed model categories.

This subject has received much attention in recent years due to new foundational work of Voevodsky, Friedlander, Suslin, and others resulting in the $\mathbf{A}^1$ homotopy theory for quasiprojective varieties over a field. Voevodsky has used this new algebraic homotopy theory to prove the Milnor conjecture (for which he was awarded the Fields Medal) and later, in collaboration with M. Rost, the full Bloch-Kato conjecture.

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External links

- An abstract for a talk on the proof of the full Bloch-Kato conjecture [1]

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[1] http://www.math.neu.edu/bhmn/suslin.html

Cohomology theory

In mathematics, specifically in algebraic topology, **cohomology** is a general term for a sequence of abelian groups defined from a co-chain complex. That is, cohomology is defined as the abstract study of **cochains**, cocycles, and coboundaries. Cohomology can be viewed as a method of assigning algebraic invariants to a topological space that has a more refined algebraic structure than does homology. Cohomology arises from the algebraic dualization of the construction of homology. In less abstract language, cochains in the fundamental sense should assign 'quantities' to the *chains* of homology theory.

From its beginning in topology, this idea became a dominant method in the mathematics of the second half of the twentieth century; from the initial idea of *homology* as a topologically invariant relation on *chains*, the range of applications of homology and cohomology theories has spread out over geometry and abstract algebra. The terminology tends to mask the fact that in many applications *cohomology*, a contravariant theory, is more natural than *homology*. At a basic level this has to do with functions and pullbacks in geometric situations: given spaces X and Y, and some kind of function F on Y, for any mapping $f\colon X \to Y$ composition with f gives rise to a function $F \circ f$ on X. Cohomology groups often also have a natural product, the cup product, which gives them a ring structure. Because of this feature, cohomology is a stronger invariant than homology, as it can differentiate between certain algebraic objects that homology cannot.

Definition

For a topological space X, the **cohomology group** $H^n(X;G)$, with coefficents in G, is defined to be the quotient Ker(δ)/Im(δ) at $C^n(X;G)$ in the cochain complex

$$\ldots \leftarrow C^{n+1}(X;G) \xleftarrow{\delta} C^n(X;G) \leftarrow \ldots \leftarrow C^0(X;G) \leftarrow 0.$$

Elements in Ker(δ) are **cocycles** and elements in Im(δ) are **coboundaries**.

History

Although cohomology is fundamental to modern algebraic topology, its importance was not seen for some 40 years after the development of homology. The concept of *dual cell structure*, which Henri Poincaré used in his proof of his Poincaré duality theorem, contained the germ of the idea of cohomology, but this was not seen until later.

There were various precursors to cohomology. In the mid-1920s, J.W. Alexander and Solomon Lefschetz founded the intersection theory of cycles on manifolds. On an n-dimensional manifold M, a p-cycle and a q-cycle with nonempty intersection will, if in general position, have intersection a $(p+q-n)$-cycle. This enables us to define a multiplication of homology classes

$$H_p(M) \times H_q(M) \to H_{p+q-n}(M).$$

Alexander had by 1930 defined a first cochain notion, based on a p-cochain on a space X having relevance to the small neighborhoods of the diagonal in X^{p+1}.

In 1931, Georges de Rham related homology and exterior differential forms, proving De Rham's theorem. This result is now understood to be more naturally interpreted in terms of cohomology.

In 1934, Lev Pontryagin proved the Pontryagin duality theorem; a result on topological groups. This (in rather special cases) provided an interpretation of Poincaré duality and Alexander duality in terms of group characters.

At a 1935 conference in Moscow, Andrey Kolmogorov and Alexander both introduced cohomology and tried to construct a cohomology product structure.

In 1936 Norman Steenrod published a paper constructing Čech cohomology by dualizing Čech homology.

From 1936 to 1938, Hassler Whitney and Eduard Čech developed the cup product (making cohomology into a graded ring) and cap product, and realized that Poincaré duality can be stated in terms of the cap product. Their theory was still limited to finite cell complexes.

In 1944, Samuel Eilenberg overcame the technical limitations, and gave the modern definition of singular homology and cohomology.

In 1945, Eilenberg and Steenrod stated the axioms defining a homology or cohomology theory. In their 1952 book, *Foundations of Algebraic Topology*, they proved that the existing homology and cohomology theories did indeed satisfy their axioms.[1]

In 1948 Edwin Spanier, building on work of Alexander and Kolmogorov, developed Alexander-Spanier cohomology.

Cohomology theories

Eilenberg-Steenrod theories

A *cohomology theory* is a family of contravariant functors from the category of pairs of topological spaces and continuous functions (or some subcategory thereof such as the category of CW complexes) to the category of Abelian groups and group homomorphisms that satisfies the Eilenberg-Steenrod axioms.

Some cohomology theories in this sense are:

- simplicial cohomology
- singular cohomology
- de Rham cohomology
- Čech cohomology
- sheaf cohomology.

Generalized cohomology theories

When one axiom (the *dimension axiom*) is relaxed, one obtains the idea of **generalized cohomology theory** or **extraordinary cohomology theory**; this allows theories based on K-theory and cobordism theory. There are others, coming from stable homotopy theory. In this context, singular homology is referred to as **ordinary homology.**

A generalized cohomology theory is "determined by its values on a point", in the sense that if one has a space given by contractible spaces (homotopy equivalent to a point), glued together along contractible spaces, as in a simplicial complex, then the cohomology of the space is determined by the cohomology of a point and the combinatorics of the patching, and effectively computable. Formally, this is computed by the excision theorem, or equivalently the Mayer–Vietoris sequence. Thus the cohomology of a point is a fundamental calculation for any generalized cohomology theory, though the cohomology of particular spaces is also of interest.

Other cohomology theories

Theories in a broader sense of *cohomology* include:[2]

- André–Quillen cohomology
- BRST cohomology
- Bonar-Claven cohomology
- Bounded cohomology
- Coherent cohomology
- Crystalline cohomology
- Cyclic cohomology
- Deligne cohomology

- Dirac cohomology
- Étale cohomology
- Flat cohomology
- Galois cohomology
- Gel'fand-Fuks cohomology
- Group cohomology
- Harrison cohomology
- Hochschild cohomology
- Intersection cohomology
- Lie algebra cohomology
- Local cohomology
- Motivic cohomology
- Non-abelian cohomology
- Perverse cohomology
- Quantum cohomology
- Schur cohomology
- Spencer cohomology
- Topological André-Quillen cohomology
- Topological Cyclic cohomology
- Topological Hochschild cohomology
- Γ cohomology

Notes

[1] Spanier, E. H. (2000) "Book reviews: Foundations of Algebraic Topology" *Bulletin of the American Mathematical Society* 37(1): pp. 114-115
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K-theory

In mathematics, **K-theory** is a tool used in several disciplines. In algebraic topology, it is an extraordinary cohomology theory known as topological K-theory. In algebra and algebraic geometry, it is referred to as algebraic K-theory. It also has some applications in operator algebras. It leads to the construction of families of *K*-functors, which contain useful but often hard-to-compute information.

In physics, K-theory and in particular twisted K-theory have appeared in Type II string theory where it has been conjectured that they classify D-branes, Ramond–Ramond field strengths and also certain spinors on generalized complex manifolds. For details, see also K-theory (physics).

Early history

The subject can be said to begin with Alexander Grothendieck (1957), who used it to formulate his Grothendieck–Riemann–Roch theorem. It takes its name from the German "Klasse", meaning "class".[1] Grothendieck needed to work with coherent sheaves on an algebraic variety *X*. Rather than working directly with the sheaves, he defined a group using (isomorphism classes of) sheaves as generators, subject to a relation that identifies any extension of two sheaves with their sum. The resulting group is called $K(X)$ when only locally free sheaves are used, or $G(X)$ when all coherent sheaves. Either of these two constructions is referred to as the Grothendieck group; $K(X)$ has cohomological behavior and $G(X)$ has homological behavior.
If X is a smooth variety, the two groups are the same. If it is a smooth affine variety, then all extensions of locally free sheaves split, so group has an alternative definition.

In topology, by applying the same construction to vector bundles, Michael Atiyah and Friedrich Hirzebruch defined $K(X)$ for a topological space X in 1959, and using the Bott periodicity theorem they made it the basis of an extraordinary cohomology theory. It played a major role in the second proof of the Index Theorem (circa 1962). Furthermore this approach led to a noncommutative K -theory for C*-algebras.
Already in 1955, Jean-Pierre Serre had used the analogy of vector bundles with projective modules to formulate Serre's conjecture, which states that projective modules over the ring of polynomials over a field are free modules; this assertion is correct, but was not settled until 20 years later. (Swan's theorem is another aspect of this analogy.) In 1959, Serre formed the Grothendieck group construction for rings, and used it to prove a weak form of the conjecture. This application was one of the beginnings of **algebraic K-theory**.

Developments

The other historical origin of algebraic K-theory was the work of Whitehead and others on what later became known as Whitehead torsion.

There followed a period in which there were various partial definitions of *higher K-theory functors*. Finally, two useful and equivalent definitions were given by Daniel Quillen using homotopy theory in 1969 and 1972. A variant was also given by Friedhelm Waldhausen in order to study the *algebraic K-theory of spaces,* which is related to the study of pseudo-isotopies. Much modern research on higher K-theory is related to algebraic geometry and the study of motivic cohomology.

The corresponding constructions involving an auxiliary quadratic form received the general name L-theory. It is a major tool of surgery theory.

In string theory the K-theory classification of Ramond–Ramond field strengths and the charges of stable D-branes was first proposed in 1997.[2]

Notes

[1] http://arxiv.org/abs/math/0602082

[2] by Ruben Minasian (http://string.lpthe.jussieu.fr/members.pl?key=7), and Gregory Moore (http://www.physics.rutgers.edu/~gmoore) in K-theory and Ramond–Ramond Charge (http://xxx.lanl.gov/abs/hep-th/9710230).

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Algebraic K-theory

In mathematics, **algebraic K-theory** is an important part of homological algebra concerned with defining and applying a sequence

$$K_n(R)$$

of functors from rings to abelian groups, for all integers *n*. For historical reasons, the **lower K-groups** K_0 and K_1 are thought of in somewhat different terms from the **higher algebraic K-groups** K_n for $n \geq 2$. Indeed, the lower groups are more accessible, and have more applications, than the higher groups. The theory of the higher K-groups is noticeably deeper, and certainly much harder to compute (even when *R* is the ring of integers).

The group $K_0(R)$ generalises the construction of the ideal class group of a ring, using projective modules. Its development in the 1960s and 1970s was linked to attempts to solve a conjecture of Serre on projective modules that now is the Quillen-Suslin theorem; numerous other connections with classical algebraic problems were found in this era. Similarly, $K_1(R)$ is a modification of the group of units in a ring, using elementary matrix theory. The group $K_1(R)$ is important in topology, especially when R is a group ring, because its quotient the Whitehead group contains the Whitehead torsion used to study problems in simple homotopy theory and surgery theory; the group $K_0(R)$ also contains other invariants such as the finiteness invariant. Since the 1980s, algebraic K-theory has increasingly had applications to algebraic geometry. For example, motivic cohomology is closely related to algebraic *K*-theory.

History

Alexander Grothendieck discovered K-theory in the mid-1950s as a framework to state his far-reaching generalization of the Riemann-Roch theorem. Within a few years, its topological counterpart was considered by Michael Atiyah and Hirzebruch and is now known as topological K-theory.

Applications of K-groups were found from 1960 onwards in surgery theory for manifolds, in particular; and numerous other connections with classical algebraic problems were brought out.

A little later a branch of the theory for operator algebras was fruitfully developed, resulting in operator K-theory and KK-theory. It also became clear that K-theory could play a role in algebraic cycle theory in algebraic geometry (Gersten's conjecture): here the *higher* K-groups become connected with the *higher codimension* phenomena, which are exactly those that are harder to access. The problem was that the definitions were lacking (or, too many and not obviously consistent). Using work of Robert Steinberg on universal central extensions of classical algebraic groups, John Milnor defined the group $K_2(A)$ of a ring A as the center, isomorphic to $H_2(E(A),Z)$, of the universal central extension of the group E(A) of infinite elementary matrices over A. (Definitions below.) There is a natural bilinear pairing from $K_1(A) \times K_1(A)$ to $K_2(A)$. In the special case of a field k, with $K_1(k)$ isomorphic to the multiplicative group GL(1,k), computations of Hideya Matsumoto showed that $K_2(k)$ is isomorphic to the group generated by $K_1(A) \times K_1(A)$ modulo an easily described set of relations.

Eventually the foundational difficulties were resolved (leaving a deep and difficult theory) by Daniel Quillen, who gave several definitions of $K_n(A)$ for arbitrary non-negative n, via the +-construction and the *Q*-construction.

Lower K-groups

The lower K-groups were discovered first, and given various ad hoc descriptions, which remain useful. Throughout, let A be a ring.

K_0

The functor K_0 takes a ring A to Grothendieck group of the set of isomorphism classes of its finitely generated projective modules, regarded as a monoid under direct sum. Any ring homomorphism $A \to B$ gives a map $K_0(A)$ by mapping (the class of) a projective A-module M to $M \otimes_A B$, making K_0 a covariant functor.

(Projective) modules over a field k are vector spaces and $K_0(k)$ is isomorphic to **Z**, by dimension. For A a Dedekind ring,

$$K_0(A) = \mathrm{Pic}(A) \oplus \mathbf{Z},$$

where Pic(A) is the Picard group of A, and similarly the reduced K-theory is given by

$$\tilde{K}_0(A) = \mathrm{Pic} A.$$

An algebro-geometric variant of this construction is applied to the category of algebraic varieties; it associates with a given algebraic variety X the Grothendieck's K-group of the category of locally free sheaves (or coherent sheaves) on X. Given a compact topological space X, the topological K-theory $K^{\mathrm{top}}(X)$ of (real) vector bundles over X coincides with K_0 of the ring of continuous real-valued functions on X. [1]

K_1

Hyman Bass provided this definition, which generalizes the group of units of a field: $K_1(A)$ is the abelianization of the infinite general linear group:

$$K_1(A) = \mathrm{GL}(A)^{\mathrm{ab}} = \mathrm{GL}(A)/[\mathrm{GL}(A), \mathrm{GL}(A)]$$

Here

$$\mathrm{GL}(A) = \mathrm{colim}\mathrm{GL}_n(A)$$

is the direct limit of the GL_n, which embeds in GL_{n+1} as the upper left block matrix, and the commutator subgroup agrees with the group generated by elementary matrices $\mathrm{E}(A) = [\mathrm{GL}(A), \mathrm{GL}(A)]$, by Whitehead's lemma. Indeed, the group $\mathrm{GL}(A)/\mathrm{E}(A)$ was first defined and studied by Whitehead,[2] and is called the **Whitehead group** of the *ring*[3] A.

As $\mathrm{E}(A) \lhd \mathrm{SL}(A)$, one can also define the **special Whitehead group** $SK_1(A) := \mathrm{SL}(A)/\mathrm{E}(A)$.

Commutative rings and fields

For A a commutative ring, one can define a determinant $\det\colon \mathrm{GL}(A) \to A^*$ to the group of units of A, which vanishes on $\mathrm{E}(A)$ and thus descends to a map $\det\colon K_1(A) \to A^*$. This map splits via the map $A^* \xrightarrow{\sim} GL_1(A) \to K_1(A)$ (unit in the upper left corner), and hence is onto, and has the special Whitehead group as kernel, yielding the split short exact sequence:

$$1 \to SK_1(A) \to K_1(A) \to A^* \to 1,$$

which is a quotient of the usual split short exact sequence defining the special linear group, namely

$$1 \to \mathrm{SL}(A) \to \mathrm{GL}(A) \to A^* \to 1.$$

Thus, since the groups in question are abelian, $K_1(A)$ splits as the direct sum of the group of units and the special Whitehead group: $K_1(A) \approx A^* \oplus SK_1(A)$.

When A is a Euclidean domain (e.g. a field, or the integers) $SK_1(A)$ vanishes, and the determinant map is an isomorphism. In particular, $\det\colon K_1(F) \xrightarrow{\sim} F^*$. This is *false* in general for PIDs, thus providing one of the rare mathematical features of Euclidean domains that do not generalize to all PIDs. An explicit PID A such that $SK_1(A)$ is nonzero was given by Grayson in 1981. A hard theorem of Bass, Milnor, and Serre shows $SK_1(A)$ vanishes when A is the ring of S-integers in any global field.

For a non-commutative ring, the determinant cannot be defined, but the map $\mathrm{GL}(A) \to K_1(A)$ generalizes the determinant.

K_2

John Milnor found the right definition of K_2: it is the center of the Steinberg group $\mathrm{St}(A)$ of A.
It can also be defined as the kernel of the map

$$\varphi\colon \mathrm{St}(A) \to \mathrm{GL}(A),$$

or as the Schur multiplier of the group of elementary matrices.

For a field k one has

$$K_2(k) = k^\times \otimes_{\mathbf{Z}} k^\times / \langle a \otimes (1-a) \mid a \neq 0, 1 \rangle.$$

Milnor *K*-theory

The above expression for K_2 of a field k led Milnor to the following definition of "higher" K-groups by

$$K_*^M(k) := T^* k^\times / (a \otimes (1-a)),$$

thus as graded parts of a quotient of the tensor algebra of the multiplicative group $k^\times$ by the two-sided ideal, generated by the

$$a \otimes (1-a)$$

for $a \neq 0,1$. For $n = 0,1,2$ these coincide with those below, but for n3 they differ in general.[4] For example, we have $K_n^M(\mathbb{F}_q) = 0$ for n3. Milnor K-theory modulo 2 is related to etale (or Galois) cohomology of the field by the Milnor conjecture, proven by Voevodsky.[5]

Higher *K*-theory

The master, definitive definitions of *K*-theory were given by Daniel Quillen, after an extended period in which uncertainty had reigned.

The +-construction

One possible definition of higher algebraic *K*-theory of rings was given by Quillen

$$K_n(R) = \pi_n(BGL(R)^+),$$

Here π_n is a homotopy group, $GL(R)$ is the direct limit of the general linear groups over R for the size of the matrix tending to infinity, B is the classifying space construction of homotopy theory, and the $^+$ is Quillen's plus construction.

This definition only holds for $n>0$ so one often defines the higher algebraic *K*-theory via

$$K_n(R) = \pi_n(BGL(R)^+ \times K_0(R))$$

Since $BGL(R)^+$ is path connected and $K_0(R)$ discrete, this definition doesn't differ in higher degrees and also holds for $n=0$.

The Q-construction

The Q-construction gives the same results as the +-construction, but it applies in more general situations. Moreover, the definition is more direct in the sense that the *K*-groups, defined via the Q-construction are functorial by definition. This fact is not automatic in the +-construction.

Suppose *P* is an exact category; associated to *P* a new category Q*P* is defined, objects of which are those of *P* and morphisms from *M′* to *M″* are isomorphism classes of diagrams

$$M' \longleftarrow N \longrightarrow M'',$$

where the first arrow is an admissible epimorphism and the second arrow is an admissible monomorphism.

The *i*-th ***K*-group** of *P* is then defined as

$$K_i(P) = \pi_{i+1}(\mathrm{BQ}P, 0)$$

with a fixed zero-object 0, where B*Q* is the *classifying space* of *Q*, which is defined to be the geometric realisation of the *nerve* of *Q*.

This definition coincides with the above definitions of K_0, K_1 and K_2.

The *K*-groups $K_i(A)$ of the ring *A* are then the *K*-groups $K_i(P_A)$ where P_A is the category of finitely generated projective *A*-modules. More generally, for a scheme *X*, the higher *K*-groups of *X* are by definition the *K*-groups of (the exact category of) locally free coherent sheaves on *X*.
The following variant of this is also used: instead of finitely generated projective (=locally free) modules, take finitely generated modules. The resulting *K*-groups are usually called *G*-groups, or *higher G-theory*. When *A* is a noetherian regular ring, then *G*- and *K*-theory coincide. Indeed, the global dimension of regular local rings is finite, i.e. any finitely generated module has a finite projective resolution, so the canonical map $K_0 \to G_0$ is surjective. It is also injective, as can be shown. This isomorphism extends to the higher *K*-groups, too.

The S-construction

A third construction of *K*-theory groups is the S-construction, due to Waldhausen.[6] It applies to categories with cofibrations (also called Waldhausen categories). This is a more general concept than exact categories.

Examples

While the Quillen algebraic K-theory has provided deep insight into various aspects of algebraic geometry and topology, the K-groups have proved particularly difficult to compute except in a few isolated but interesting cases.

Algebraic K-groups of finite fields

The first and one of the most important calculations of the higher algebraic K-groups of a ring were made by Quillen himself for the case of finite fields:

Theorem. Let *F* be a finite field with *q* elements. Then

$$K_0(F) = Z \text{ , } K_{2i}(F) = 0$$

for $i \neq 0$, and

$$K_{2i-1}(F) = \mu_{q^i-1} \text{ for } i = 1, 2, \dots$$

where μ_r denotes the cyclic group with *r* elements.

Algebraic K-groups of rings of integers

Quillen proved that if *A* is the ring of algebraic integers in an algebraic number field *F* (a finite extension of the rationals), then the algebraic K-groups of *A* are finitely generated. Borel used this to calculate $K_i(A)$ and $K_i(F)$ modulo torsion. For example, for the integers **Z**, Borel proved that (modulo torsion)

$K_i(\mathbf{Z}) = 0$ for positive *i* unless $i = 4k+1$ with *k* positive

and (modulo torsion)

$K_{4k+1}(\mathbf{Z}) = \mathbf{Z}$ for positive *k*.

The torsion subgroups of $K_{2i+1}(\mathbf{Z})$, and the orders of the finite groups $K_{4k+2}(\mathbf{Z})$ have recently been determined, but whether the latter groups are cyclic, and whether the groups $K_{4k}(\mathbf{Z})$ vanish depends upon Vandiver's conjecture about the class groups of cyclotomic integers.

Applications and open questions

Algebraic K-groups are used in conjectures on special values of L-functions and the formulation of an non-commutative main conjecture of Iwasawa theory and in construction of higher regulators.

Another fundamental conjecture due to Hyman Bass (Bass conjecture) says that all *G*-groups *G*(*A*) (that is to say, *K*-groups of the category of finitely generated *A*-modules) are finitely generated when *A* is a finitely generated **Z**-algebra.[7]

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[2] J.H.C. Whitehead, *Simple homotopy types* Amer. J. Math. , 72 (1950) pp. 1–57

[3] Not to be confused with the Whitehead group of a *group*.

[4] (Weibel 2005), cf. Lemma 1.8

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[7] (Friedlander & Weibel 1999), Lecture VI

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External links

- C. Weibel " The K-book: An introduction to algebraic K-theory (http://www.math.rutgers.edu/~weibel/Kbook.html)"

Topological K-theory

In mathematics, **topological K-theory** is a branch of algebraic topology. It was founded to study vector bundles on general topological spaces, by means of ideas now recognised as (general) K-theory that were introduced by Alexander Grothendieck. The early work on topological K-theory is due to Michael Atiyah and Friedrich Hirzebruch.

Definitions

Let *X* be a compact Hausdorff space and $k = \mathbb{R}$ or $k = \mathbb{C}$. Then $K_k(X)$ is the Grothendieck group of the commutative monoid whose elements are the isomorphism classes of finite dimensional k -vector bundles on *X* with the operation

$$[E] \oplus [F] = [E \oplus F]$$

for vector bundles *E*, *F*. Usually, $K_k(X)$ is denoted $KO(X)$ in real case and $KU(X)$ in the complex case.

More explicitly, **stable equivalence**, the equivalence relation on bundles *E* and *F* on *X* of defining the same element in *K*(*X*), occurs when there is a trivial bundle *G*, so that

$$E \oplus G \cong F \oplus G .$$

Under the tensor product of vector bundles *K*(*X*) then becomes a commutative ring.

The rank of a vector bundle carries over to the K-group. Define the homomorphism

$$K(X) \to \check{H}^0(X, \mathbb{Z})$$

where $\check{H}^0(X, \mathbb{Z})$ is the 0-group of Čech cohomology which is equal to the group of locally constant functions with values in $\mathbb{Z}$.

If X has a distinguished basepoint x_0, then the reduced K-group (cf. reduced homology) satisfies

$$K(X) \cong \tilde{K}(X) \oplus K(\{x_0\})$$

and is defined as either the kernel of $K(X) \to K(\{x_0\})$(where $\{x_0\} \to X$ is basepoint inclusion) or the cokernel of $K(\{x_0\}) \to K(X)$(where $X \to \{x_0\}$ is the constant map).

When X is a connected space, $\tilde{K}(X) \cong \mathrm{Ker}(K(X) \to \check{H}^0(X, \mathbb{Z}) = \mathbb{Z})$.

The definition of the functor K extends to the category of pairs of compact spaces (in this category, an object is a pair (X, Y), where X is compact and $Y \subset X$ is closed, a morphism between (X, Y) and (X', Y') is a continuous map $f : X \to X'$ such that $f(Y) \subset Y'$)

$$K(X, Y) := \tilde{K}(X/Y).$$

The reduced K-group is given by $x_0 = \{Y\}$.

The definition

$$K^n_{\mathbb{C}}(X, Y) = \tilde{K}_{\mathbb{C}}(S^{|n|}(X/Y))$$

gives the sequence of K-groups for $n \in \mathbb{Z}$, where *S* denotes the reduced suspension.

Properties

- K^n is a contravariant functor.
- The classifying space of $\tilde{K}$ is BO_k (BO, in real case; BU in complex case), i.e. $\tilde{K}_k(X) \cong [X, BO_k]$.
- The classifying space of K is $\mathbb{Z} \times BO_k$ ($\mathbb{Z}$ with discrete topology), i.e. $K_k(X) \cong [X, \mathbb{Z} \times BO_k]$.
- There is a natural ring homomorphism $K^*(X) \to H^{2*}(X, \mathbb{Q})$, the Chern character, such that $K^*(X) \otimes \mathbb{Q} \to H^{2*}(X, \mathbb{Q})$ is an isomorphism.
- Topological K-theory can be generalized vastly to a functor on C*-algebras, see operator K-theory and KK-theory.

Bott periodicity

The phenomenon of periodicity named for Raoul Bott (see Bott periodicity theorem) can be formulated this way:

- $K(X \times S^2) = K(X) \otimes K(S^2),$ and $K(S^2) = \mathbb{Z}[H]/(H-1)^2;$ where H is the class of the tautological bundle on the $S^2 = \mathbb{C}P^1$, i.e. the Riemann sphere as complex projective line.
- $\tilde{K}^{n+2}(X) = \tilde{K}^n(X).$
- $\Omega^2\mathrm{BU} \simeq \mathrm{BU} \times \mathbf{Z}.$

In real K-theory there is a similar periodicity, but *modulo* 8.

Notes

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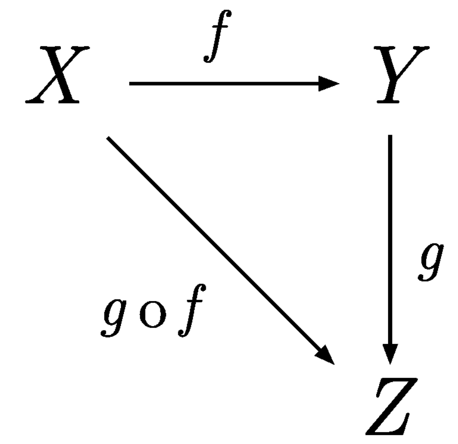
Category Theories

Category theory



A category with objects X, Y, Z and morphisms *f*, *g*

Category theory is an area of study in mathematics that examines in an abstract way the properties of particular mathematical concepts, by formalising them as collections of *objects* and *arrows* (also called morphisms, although this term also has a specific, non category-theoretical sense), where these collections satisfy certain basic conditions. Many significant areas of mathematics can be formalised as categories, and the use of category theory allows many intricate and subtle mathematical results in these fields to be stated, and proved, in a much simpler way than without the use of categories.

The most accessible example of a category is the Category of sets, where the objects are sets and the arrows are functions from one set to another. However it is important to note that the objects of a category need not be sets nor the arrows functions; any way of formalising a mathematical concept such that it meets the basic conditions on the behaviour of objects and arrows is a valid category, and all the results of category theory will apply to it.

One of the simplest examples of a category (which is a very important concept in topology) is that of groupoid, defined as a category whose arrows or morphisms are all invertible. Categories now appear in most branches of mathematics, some areas of theoretical computer science where they correspond to types, and mathematical physics where they can be used to describe vector spaces. Categories were first introduced by Samuel Eilenberg and Saunders Mac Lane in 1942–45, in connection with algebraic topology.

Category theory has several faces known not just to specialists, but to other mathematicians. A term dating from the 1940s, "general abstract nonsense", refers to its high level of abstraction, compared to more classical branches of mathematics. Homological algebra is category theory in its aspect of organising and suggesting manipulations in abstract algebra. Diagram chasing is a visual method of arguing with abstract "arrows" joined in diagrams. Note that arrows between categories are called functors, subject to specific defining commutativity conditions; moreover, categorical diagrams and sequences can be defined as functors (viz. Mitchell, 1965). An arrow between two functors is a natural transformation when it is subject to certain naturality or commutativity conditions. Functors and natural transformations ('naturality') are the key concepts in category theory[1] . Topos theory is a form of abstract sheaf theory, with geometric origins, and leads to ideas such as pointless topology. A topos can also be considered as a specific type of category with two additional topos axioms.

Background

The study of categories is an attempt to *axiomatically* capture what is commonly found in various classes of related *mathematical structures* by relating them to the *structure-preserving functions* between them. A systematic study of category theory then allows us to prove general results about any of these types of mathematical structures from the axioms of a category.

Consider the following example. The class **Grp** of groups consists of all objects having a "group structure". One can proceed to prove theorems about groups by making logical deductions from the set of axioms. For example, it is immediately proved from the axioms that the identity element of a group is unique.

Instead of focusing merely on the individual objects (e.g., groups) possessing a given structure, category theory emphasizes the morphisms – the structure-preserving mappings – *between* these objects; by studying these morphisms, we are able to learn more about the structure of the objects. In the case of groups, the morphisms are the group homomorphisms. A group homomorphism between two groups "preserves the group structure" in a precise sense – it is a "process" taking one group to another, in a way that carries along information about the structure of the first group into the second group. The study of group homomorphisms then provides a tool for studying general properties of groups and consequences of the group axioms.

A similar type of investigation occurs in many mathematical theories, such as the study of continuous maps (morphisms) between topological spaces in topology (the associated category is called **Top**), and the study of smooth functions (morphisms) in manifold theory.

If one axiomatizes relations instead of functions, one obtains the theory of allegories.

Functors

Abstracting again, a category is *itself* a type of mathematical structure, so we can look for "processes" which preserve this structure in some sense; such a process is called a functor. A functor associates to every object of one category an object of another category, and to every morphism in the first category a morphism in the second.

In fact, what we have done is define a category *of categories and functors* – the objects are categories, and the morphisms (between categories) are functors.

By studying categories and functors, we are not just studying a class of mathematical structures and the morphisms between them; we are studying the *relationships between various classes of mathematical structures*. This is a fundamental idea, which first surfaced in algebraic topology. Difficult *topological* questions can be translated into *algebraic* questions which are often easier to solve. Basic constructions, such as the fundamental group or fundamental groupoid [2] of a topological space, can be expressed as fundamental functors [2] to the category of groupoids in this way, and the concept is pervasive in algebra and its applications.

Natural transformation

Abstracting yet again, constructions are often "naturally related" – a vague notion, at first sight. This leads to the clarifying concept of natural transformation, a way to "map" one functor to another. Many important constructions in mathematics can be studied in this context. "Naturality" is a principle, like general covariance in physics, that cuts deeper than is initially apparent.

Historical notes

In 1942–45, Samuel Eilenberg and Saunders Mac Lane introduced categories, functors, and natural transformations as part of their work in topology, especially algebraic topology. Their work was an important part of the transition from intuitive and geometric homology to axiomatic homology theory. Eilenberg and Mac Lane later wrote that their goal was to understand natural transformations; in order to do that, functors had to be defined, which required categories.

Stanisław Ulam, and some writing on his behalf, have claimed that related ideas were current in the late 1930s in Poland. Eilenberg was Polish, and studied mathematics in Poland in the 1930s. Category theory is also, in some sense, a continuation of the work of Emmy Noether (one of Mac Lane's teachers) in formalizing abstract processes; Noether realized that in order to understand a type of mathematical structure, one needs to understand the processes preserving that structure. In order to achieve this understanding, Eilenberg and Mac Lane proposed an axiomatic formalization of the relation between structures and the processes preserving them.

The subsequent development of category theory was powered first by the computational needs of homological algebra, and later by the axiomatic needs of algebraic geometry, the field most resistant to being grounded in either axiomatic set theory or the Russell-Whitehead view of united foundations. General category theory, an extension of universal algebra having many new features allowing for semantic flexibility and higher-order logic, came later; it is now applied throughout mathematics.

Certain categories called topoi (singular *topos*) can even serve as an alternative to axiomatic set theory as a foundation of mathematics. These foundational applications of category theory have been worked out in fair detail as a basis for, and justification of, constructive mathematics. More recent efforts to introduce undergraduates to categories as a foundation for mathematics include Lawvere and Rosebrugh (2003) and Lawvere and Schanuel (1997).

Categorical logic is now a well-defined field based on type theory for intuitionistic logics, with applications in functional programming and domain theory, where a cartesian closed category is taken as a non-syntactic description of a lambda calculus. At the very least, category theoretic language clarifies what exactly these related areas have in common (in some abstract sense).

Categories, objects, and morphisms

A *category* C consists of the following three mathematical entities:

- A class ob(C), whose elements are called *objects*;
- A class hom(C), whose elements are called morphisms or maps or *arrows*. Each morphism f has a unique *source object* a and *target object* b. We write f: $a \to b$, and we say "f is a morphism from a to b". We write hom(a, b) (or Hom(a, b), or $\hom_C(a, b)$, or Mor(a, b), or $C(a, b)$) to denote the *hom-class* of all morphisms from a to b.
- A binary operation $\circ$, called *composition of morphisms*, such that for any three objects a, b, and c, we have hom(a, b) × hom(b, c) → hom(a, c). The composition of f: $a \to b$ and g: $b \to c$ is written as $g \circ f$ or gf[3], governed by two axioms:
  - Associativity: If $f : a \to b$, $g : b \to c$ and $h : c \to d$ then $h \circ (g \circ f) = (h \circ g) \circ f$, and
  - Identity: For every object x, there exists a morphism $1_x : x \to x$ called the *identity morphism for x*, such that for every morphism $f : a \to b$, we have $1_b \circ f = f = f \circ 1_a$.

From these axioms, it can be proved that there is exactly one identity morphism for every object. Some authors deviate from the definition just given by identifying each object with its identity morphism.

Relations among morphisms (such as $fg = h$) are often depicted using commutative diagrams, with "points" (corners) representing objects and "arrows" representing morphisms.

Properties of morphisms

Some morphisms have important properties. A morphism $f : a \to b$ is:

- a monomorphism (or *monic*) if $f \circ g_1 = f \circ g_2$ implies $g_1 = g_2$ for all morphisms $g_1, g_2 : x \to a$.
- an epimorphism (or *epic*) if $g_1 \circ f = g_2 \circ f$ implies $g_1 = g_2$ for all morphisms $g_1, g_2 : b \to x$.
- an isomorphism if there exists a morphism $g : b \to a$ with $f \circ g = 1_b$ and $g \circ f = 1_a$.[4]
- an endomorphism if $a = b$. end(a) denotes the class of endomorphisms of a.
- an automorphism if f is both an endomorphism and an isomorphism. aut(a) denotes the class of automorphisms of a.

Functors

Functors are structure-preserving maps between categories. They can be thought of as morphisms in the category of all (small) categories.

A (**covariant**) functor F from a category C to a category D, written $F : C \to D$, consists of:

- for each object x in C, an object $F(x)$ in D; and
- for each morphism $f : x \to y$ in C, a morphism $F(f) : F(x) \to F(y)$,

such that the following two properties hold:

- For every object x in C, $F(1_x) = 1_{F(x)}$;
- For all morphisms $f : x \to y$ and $g : y \to z$, $F(g \circ f) = F(g) \circ F(f)$.

A **contravariant** functor F: $C \to D$, is like a covariant functor, except that it "turns morphisms around" ("reverses all the arrows"). More specifically, every morphism $f : x \to y$ in C must be assigned to a morphism $F(f) : F(y) \to F(x)$ in D. In other words, a contravariant functor is a covariant functor from the opposite category C^{op} to D.

Natural transformations and isomorphisms

A *natural transformation* is a relation between two functors. Functors often describe "natural constructions" and natural transformations then describe "natural homomorphisms" between two such constructions. Sometimes two quite different constructions yield "the same" result; this is expressed by a natural isomorphism between the two functors.

If F and G are (covariant) functors between the categories C and D, then a natural transformation η from F to G associates to every object x in C a morphism $\eta_x : F(x) \to G(x)$ in D such that for every morphism $f : x \to y$ in C, we have $\eta_y \circ F(f) = G(f) \circ \eta_x$; this means that the following diagram is commutative:

$$\begin{tikzcd} F(X) \arrow[r, "F(f)"] \arrow[d, "\eta_X"'] & F(Y) \arrow[d, "\eta_Y"] \\ G(X) \arrow[r, "G(f)"'] & G(Y) \end{tikzcd}$$

The two functors F and G are called *naturally isomorphic* if there exists a natural transformation from F to G such that η_x is an isomorphism for every object x in C.

Universal constructions, limits, and colimits

Using the language of category theory, many areas of mathematical study can be cast into appropriate categories, such as the categories of all sets, groups, topologies, and so on. These categories surely have some objects that are "special" in a certain way, such as the empty set or the product of two topologies, yet in the definition of a category, objects are considered to be atomic, i.e., we *do not know* whether an object A is a set, a topology, or any other abstract concept – hence, the challenge is to define special objects without referring to the internal structure of those objects. But how can we define the empty set without referring to elements, or the product topology without referring to open sets?

The solution is to characterize these objects in terms of their relations to other objects, as given by the morphisms of the respective categories. Thus, the task is to find *universal properties* that uniquely determine the objects of interest. Indeed, it turns out that numerous important constructions can be described in a purely categorical way. The central concept which is needed for this purpose is called categorical *limit*, and can be dualized to yield the notion of a *colimit*.

Equivalent categories

It is a natural question to ask: under which conditions can two categories be considered to be "essentially the same", in the sense that theorems about one category can readily be transformed into theorems about the other category? The major tool one employs to describe such a situation is called *equivalence of categories*, which is given by appropriate functors between two categories. Categorical equivalence has found numerous applications in mathematics.

Further concepts and results

The definitions of categories and functors provide only the very basics of categorical algebra; additional important topics are listed below. Although there are strong interrelations between all of these topics, the given order can be considered as a guideline for further reading.

- The functor category D^C has as objects the functors from C to D and as morphisms the natural transformations of such functors. The Yoneda lemma is one of the most famous basic results of category theory; it describes representable functors in functor categories.
- Duality: Every statement, theorem, or definition in category theory has a *dual* which is essentially obtained by "reversing all the arrows". If one statement is true in a category C then its dual will be true in the dual category C^{op}. This duality, which is transparent at the level of category theory, is often obscured in applications and can lead to surprising relationships.
- Adjoint functors: A functor can be left (or right) adjoint to another functor that maps in the opposite direction. Such a pair of adjoint functors typically arises from a construction defined by a universal property; this can be seen as a more abstract and powerful view on universal properties.

Higher-dimensional categories

Many of the above concepts, especially equivalence of categories, adjoint functor pairs, and functor categories, can be situated into the context of *higher-dimensional categories*. Briefly, if we consider a morphism between two objects as a "process taking us from one object to another", then higher-dimensional categories allow us to profitably generalize this by considering "higher-dimensional processes".

For example, a (strict) 2-category is a category together with "morphisms between morphisms", i.e., processes which allow us to transform one morphism into another. We can then "compose" these "bimorphisms" both horizontally and vertically, and we require a 2-dimensional "exchange law" to hold, relating the two composition laws. In this context, the standard example is **Cat**, the 2-category of all (small) categories, and in this example, bimorphisms of morphisms are simply natural transformations of morphisms in the usual sense. Another basic example is to consider a 2-category with a single object; these are essentially monoidal categories. Bicategories are a weaker notion of 2-dimensional categories in which the composition of morphisms is not strictly associative, but only associative "up to" an isomorphism.

This process can be extended for all natural numbers n, and these are called n-categories. There is even a notion of *ω-category* corresponding to the ordinal number ω.

Higher-dimensional categories are part of the broader mathematical field of higher-dimensional algebra, a concept introduced by Ronald Brown. For a conversational introduction to these ideas, see John Baez, 'A Tale of n-categories' (1996). [5]

See also

- Domain theory
- Enriched category theory
- Glossary of category theory
- Higher category theory
- Higher-dimensional algebra
- Important publications in category theory
- Timeline of category theory and related mathematics

Notes

[1] *Categories for the Working Mathematician*, 2nd Edition, p 18: "As Eilenberg-Mac Lane first observed, 'category' has been defined in order to be able to define 'functor' and 'functor' has been defined in order to be able to define 'natural transformation' ".

[2] http://planetphysics.org/encyclopedia/FundamentalGroupoidFunctor.html

[3] Some authors compose in the opposite order, writing *fg* or $f \circ g$ for $g \circ f$. Computer scientists using category theory very commonly write *f;g* for $g \circ f$

[4] Note that a morphism that is both epic and monic is not necessarily an isomorphism! For example, in the category consisting of two objects A and B, the identity morphisms, and a single morphism f from A to B, f is both epic and monic but is not an isomorphism.

[5] http://math.ucr.edu/home/baez/week73.html

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- List of academic conferences on category theory (http://www.mta.ca/~cat-dist/)
- Baez, John, 1996," The Tale of *n*-categories. (http://math.ucr.edu/home/baez/week73.html)" An informal introduction to higher order categories.
- The catsters (http://www.youtube.com/user/TheCatsters) , a Youtube channel about category theory.
- Category Theory (http://planetmath.org/?op=getobj&from=objects&id=5622) on PlanetMath
- Video archive (http://categorieslogicphysics.wikidot.com/events) of recorded talks relevant to categories, logic and the foundations of physics.
- Interactive Web page (http://www.j-paine.org/cgi-bin/webcats/webcats.php) which generates examples of categorical constructions in the category of finite sets.

Category (mathematics)

In mathematics, a **category** is an algebraic structure consisting of a collection of "objects", linked together by a collection of "arrows" that have two basic properties: the ability to compose the arrows associatively and the existence of an identity arrow for each object. Objects and arrows may be abstract entities of any kind. Categories generalize monoids, groupoids and preorders. In addition, the notion of category provides a fundamental and abstract way to describe mathematical entities and their relationships. This is the central idea of *category theory*, a branch of mathematics which seeks to generalize all of mathematics in terms of objects and arrows, independent of what the objects and arrows represent. Virtually every branch of modern mathematics can be described in terms of categories, and doing so often reveals deep insights and similarities between seemingly different areas of mathematics. For more extensive motivational background and historical notes, see category theory and the list of category theory topics.

Two categories are the same if they have the same collection of objects, the same collection of arrows, and the same associative method of composing any pair of arrows. Two categories may also be considered "equivalent" for purposes of category theory, even if they are not precisely the same. Many well-known categories are conventionally identified by a short capitalized word or abbreviation in bold or italics such as **Set** (category of sets and set functions),[1] **Ring** (category of rings and ring homomorphisms),[2] or **Top** (category of topological spaces and continuous maps).[3]

Definition

A **category** C consists of

- a class ob(C) of **objects**:
- a class hom(C) of **morphisms**, or **arrows**, or **maps**, between the objects. Each morphism f has a unique *source object* a and *target object* b where a and b are in ob(C). We write f: $a \to b$, and we say "f is a morphism from a to b". We write hom(a, b) (or $\hom_C(a, b)$ when there may be confusion about to which category hom(a, b) refers) to denote the **hom-class** of all morphisms from a to b. (Some authors write Mor(a, b) or simply $C(a, b)$ instead.)
- for every three objects a, b and c, a binary operation $\hom(a, b) \times \hom(b, c) \to \hom(a, c)$ called *composition of morphisms*; the composition of $f : a \to b$ and $g : b \to c$ is written as $g \circ f$ or gf. (Some authors write fg or $f;g$.)

such that the following axioms hold:

- (associativity) if $f : a \to b$, $g : b \to c$ and $h : c \to d$ then $h \circ (g \circ f) = (h \circ g) \circ f$, and
- (identity) for every object x, there exists a morphism $1_x : x \to x$ (some authors write id_x) called the *identity morphism for x*, such that for every morphism $f : a \to b$, we have $1_b \circ f = f = f \circ 1_a$.

From these axioms, one can prove that there is exactly one identity morphism for every object. Some authors use a slight variation of the definition in which each object is identified with the corresponding identity morphism.

History

Category theory first appeared in a paper entitled "General Theory of Natural Equivalences", written by Samuel Eilenberg and Saunders Mac Lane in 1945.[4]

Small and large categories

A category C is called **small** if both ob(C) and hom(C) are actually sets and not proper classes, and **large** otherwise. A **locally small category** is a category such that for all objects a and b, the hom-class hom(a, b) is a set, called a **homset**. Many important categories in mathematics (such as the category of sets), although not small, are at least locally small.

Examples

The class of all sets together with all functions between sets, where composition is the usual function composition, forms a large category, **Set**.[5] It is the most basic and the most commonly used category in mathematics. The category **Rel** consists of all sets, with binary relations as morphisms. Abstracting from relations instead of functions yields allegories instead of categories.

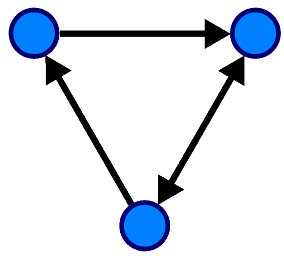
Any class can be viewed as a category whose only morphisms are the identity morphisms. Such categories are called discrete. For any given set I, the *discrete category on I* is the small category that has the elements of I as objects and only the identity morphisms as morphisms.[6] Discrete categories are the simplest kind of category.

Any preordered set (P, ≤) forms a small category, where the objects are the members of P, the morphisms are arrows pointing from x to y when $x \le y$. Between any two objects there can be at most one morphism. The existence of identity morphisms and the composability of the morphisms are guaranteed by the reflexivity and the transitivity of the preorder.[7] By the same argument, any partially ordered set and any equivalence relation can be seen as a small category. Any ordinal number can be seen as a category when viewed as a ordered set.

Any monoid (any algebraic structure with a single associative binary operation and an identity element) forms a small category with a single object x. (Here, x is any fixed set.) The morphisms from x to x are precisely the elements of the monoid, the identity morphism of x is the identity of the monoid, and the categorical composition of morphisms is given by the monoid operation.[8] Several definitions and theorems about monoids may be generalized for categories.

Any group can be seen as a category with a single object in which every morphism is invertible (for every morphism f there is a morphism g that is both left and right inverse to f under composition) by viewing the group as acting on itself by left multiplication.[9] A morphism which is invertible in this sense is called an isomorphism.

A groupoid is a category in which every morphism is an isomorphism.[10] Groupoids are generalizations of groups, group actions and equivalence relations.


A directed graph.

Any directed graph generates a small category: the objects are the vertices of the graph, and the morphisms are the paths in the graph (augmented with loops as needed) where composition of morphisms is concatenation of paths. Such a category is called the *free category* generated by the graph.

The class of all preordered sets with monotonic functions as morphisms forms a category, **Ord**. It is a concrete category, i.e. a category obtained by adding some type of structure onto **Set**, and requiring that morphisms are functions that respect this added structure.

The class of all groups with group homomorphisms as morphisms and function composition as the composition operation forms a large category, **Grp**.[11] Like **Ord**, **Grp** is a concrete category. The category **Ab**, consisting of all abelian groups and their group homomorphisms, is a full subcategory of **Grp**, and the prototype of an abelian category.[12] Other examples of concrete categories are given by the following table.

Category	Objects	Morphisms
Mag	magmas	magma homomorphisms
$\mathbf{Man}^p$	smooth manifolds	p-times continuously differentiable maps
Met	metric spaces	short maps
R-Mod	R-Modules, where R is a Ring	module homomorphisms
Ring	rings	ring homomorphisms
Set	sets	functions
Top	topological spaces	continuous functions
Uni	uniform spaces	uniformly continuous functions
$\mathbf{Vect}_K$	vector spaces over the field K	K-linear maps

Fiber bundles with bundle maps between them form a concrete category.

The category **Cat** consists of all small categories, with functors between them as morphisms.

Construction of new categories

Dual category

Any category C can itself be considered as a new category in a different way: the objects are the same as those in the original category but the arrows are those of the original category reversed. This is called the *dual* or *opposite category* and is denoted C^{op}.

Product categories

If C and D are categories, one can form the *product category* $C \times D$: the objects are pairs consisting of one object from C and one from D, and the morphisms are also pairs, consisting of one morphism in C and one in D. Such pairs can be composed componentwise.

Types of morphisms

A morphism $f : a \to b$ is called

- a *monomorphism* (or *monic*) if $fg_1 = fg_2$ implies $g_1 = g_2$ for all morphisms $g_1, g_2 : x \to a$.
- an *epimorphism* (or *epic*) if $g_1f = g_2f$ implies $g_1 = g_2$ for all morphisms $g_1, g_2 : b \to x$.
- a **bimorphism** if it is both a monomorphism and an epimorphism.
- a *retraction* if it has a right inverse, i.e. if there exists a morphism $g : b \to a$ with $fg = 1_b$.
- a *section* if it has a left inverse, i.e. if there exists a morphism $g : b \to a$ with $gf = 1_a$.
- an *isomorphism* if it has an inverse, i.e. if there exists a morphism $g : b \to a$ with $fg = 1_b$ and $gf = 1_a$.[13]
- an *endomorphism* if $a = b$. The class of endomorphisms of a is denoted end(a).
- an *automorphism* if f is both an endomorphism and an isomorphism. The class of automorphisms of a is denoted aut(a).

Every retraction is an epimorphism. Every section is a monomorphism. The following three statements are equivalent:

- f is a monomorphism and a retraction;
- f is an epimorphism and a section;
- f is an isomorphism.

Relations among morphisms (such as $fg = h$) can most conveniently be represented with commutative diagrams, where the objects are represented as points and the morphisms as arrows.[14]

Types of categories

- In many categories, e.g. **Ab** or $\mathbf{Vect}_K$, the hom-sets hom(a, b) are not just sets but actually abelian groups, and the composition of morphisms is compatible with these group structures; i.e. is bilinear. Such a category is called preadditive. If, furthermore, the category has all finite products and coproducts, it is called an additive category. If all morphisms have a kernel and a cokernel, and all epimorphisms are cokernels and all monomorphisms are kernels, then we speak of an abelian category. A typical example of an abelian category is the category of abelian groups.
- A category is called complete if all limits exist in it. The categories of sets, abelian groups and topological spaces are complete.
- A category is called cartesian closed if it has finite direct products and a morphism defined on a finite product can always be represented by a morphism defined on just one of the factors. Examples include **Set** and **CPO**, the category of complete partial orders with Scott-continuous functions.
- A topos is a certain type of cartesian closed category in which all of mathematics can be formulated (just like classically all of mathematics is formulated in the category of sets). A topos can also be used to represent a logical theory.

Notes

[1] Jacobson (2009), p. 11, ex. 1.
[2] Jacobson (2009), p. 12, ex. 9.
[3] Jacobson (2009), p. 13, ex. 13.
[4] Sica (2006), p. 223; Awodey (2006), p. 1.
[5] Jacobson (2009), p. 11, ex. 1.
[6] Jacobson (2009), p. 12, ex. 8.
[7] Jacobson (2009), p. 13, ex. 12.
[8] Jacobson (2009), p. 12, ex. 5.
[9] Jacobson (2009), p. 12, ex. 6.
[10] Jacobson (2009), p. 12, ex. 7.
[11] Jacobson (2009), p. 11, ex. 3.
[12] Jacobson (2009), p. 11, ex. 4.
[13] Jacobson (2009), p. 12.
[14] Jacobson (2009), p. 10.

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- Homepage of the Categories mailing list (http://www.mta.ca/~cat-dist/categories.html), with extensive list of resources
- *Category Theory* section of Alexandre Stefanov's list of free online mathematics resources (http://web.archive.org/web/20091027042032/http://us.geocities.com/alex_stef/mylist.html)

Glossary of category theory

This is a glossary of properties and concepts in category theory in mathematics.

Categories

A category **A** is said to be:

- **small** provided that the class of all morphisms is a set (i.e., not a proper class); otherwise **large**.
- **locally small** provided that the morphisms between every pair of objects *A* and *B* form a set.
- Some authors assume a foundation in which the collection of all classes forms a "conglomerate", in which case a **quasicategory** is a category whose objects and morphisms merely form a conglomerate.[1] (NB other authors use the term "quasicategory" with a different meaning.[2])
- **isomorphic** to a category **B** provided that there is an isomorphism between them.
- **equivalent** to a category **B** provided that there is an equivalence between them.
- **concrete** provided that there is a faithful functor from **A** to **Set**; e.g., **Vec**, **Grp** and **Top**.
- **discrete** provided that each morphism is an identity morphism (of some object).
- **thin** category provided that there is at most one morphism between any pair of objects.
- a **subcategory** of a category **B** provided that there is an inclusion functor given from **A** to **B**.
- a **full subcategory** of a category **B** provided that the inclusion functor is full.
- **wellpowered** provided for each object *A* there is only a set of pairwise non-isomorphic subobjects.
- **complete** provided that all small limits exist.
- **cartesian closed** provided that it has a terminal object and that any two objects have a product and exponential.
- **abelian** provided that it has a zero object, it has all pullbacks and pushouts, and all monomorphisms and epimorphisms are normal.
- **normal** provided that every monic is normal.[3]
- **balanced** if every bimorphism is an isomorphism.
- ***R*-linear** (*R* is a commutative ring) if **A** is locally small, each hom set is an *R*-module, and composition of morphisms is *R*-bilinear. The category **A** is also said to be **over *R***.

Morphisms

A morphism *f* in a category is called:

- an **epimorphism** provided that $g = h$ whenever $g \circ f = h \circ f$. In other words, *f* is the dual of a monomorphism.
- an **identity** provided that *f* maps an object *A* to *A* and for any morphisms *g* with domain *A* and *h* with codomain *A*, $g \circ f = g$ and $f \circ h = h$.
- an **inverse** to a morphism *g* if $g \circ f$ is defined and is equal to the identity morphism on the domain of *f*, and $f \circ g$ is defined and equal to the identity morphism on the codomain of *g*. The inverse of *g* is unique and is denoted by g^{-1}
- an **isomorphism** provided that there exists an *inverse* of *f*.
- a **monomorphism** (also called **monic**) provided that $g = h$ whenever $f \circ g = f \circ h$; e.g., an injection in **Set**. In other words, *f* is the dual of an epimorphism.

Functors

A functor *F* is said to be:

- a **constant** provided that *F* maps every object in a category to the same object *A* and every morphism to the identity on *A*.
- **faithful** provided that *F* is injective when restricted to each hom-set.
- **full** provided that *F* is surjective when restricted to each hom-set.
- **isomorphism-dense** (sometimes called **essentially surjective**) provided that for every *B* there exists *A* such that *F*(*A*) is isomorphic to *B*.
- an **equivalence** provided that *F* is faithful, full and isomorphism-dense.
- **amnestic** provided that if *k* is an isomorphism and *F*(*k*) is an identity, then *k* is an identity.
- **reflect identities** provided that if *F*(*k*) is an identity then *k* is an identity as well.
- **reflect isomorphisms** provided that if *F*(*k*) is an isomorphism then *k* is an isomorphism as well.

Objects

An object *A* in a category is said to be:

- **isomorphic** to an object B provided that there is an isomorphism between *A* and *B*.
- **initial** provided that there is exactly one morphism from *A* to each object B; e.g., empty set in **Set**.
- **terminal** provided that there is exactly one morphism from each object B to *A*; e.g., singletons in **Set**.
- a **zero object** if it is both initial and terminal, such as a trivial group in **Grp**.

An object *A* in an abelian category is:

- **simple** if it is not isomorphic to the zero object and any subobject of *A* is isomorphic to zero or to *A*.
- **finite length** if it has a composition series. The number of proper subobjects in any such composition series is called the **length** of *A*.[4]

Notes

[1] Adámek, Jiří; Herrlich, Horst, and Strecker, George E (2004) [1990] (PDF). *Abstract and Concrete Categories (The Joy of Cats)* (http://katmat.math.uni-bremen.de/acc/). New York: Wiley & Sons. p. 40. ISBN 0-471-60922-6. .
[2] Joyal, A. (2002). "Quasi-categories and Kan complexes". *Journal of Pure and Applied Algebra* **175**: 207–222. doi:10.1016/S0022-4049(02)00135-4.
[3] http://planetmath.org/encyclopedia/NormalCategory.html
[4] Kashiwara & Schapira 2006, exercise 8.20

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Dual (category theory)

In category theory, a branch of mathematics, **duality** is a correspondence between properties of a category C and so-called **dual properties** of the opposite category C^{op}. Given a statement regarding the category C, by interchanging the source and target of each morphism as well as interchanging the order of composing two morphisms, a corresponding dual statement is obtained regarding the opposite category C^{op}. **Duality**, as such, is the assertion that truth is invariant under this operation on statements. In other words, if a statement is true about C, then its dual statement is true about C^{op}. Also, if a statement is false about C, then its dual has to be false about C^{op}.

Given a concrete category C, it is often the case that the opposite category C^{op} per se is abstract. C^{op} need not be a category that arises from mathematical practice. In this case, another category D is also termed to be in **duality** with C if D and C^{op} are equivalent as categories.

In the case when C and its opposite C^{op} are equivalent, such a category is **self-dual**.

Formal definition

We define the elementary language of category theory as the two-sorted first order language with objects and morphisms as distinct sorts, together with the relations of an object being the source or target of a morphism and a symbol for composing two morphisms.

Let σ be any statement in this language. We form the dual σ^{op} as follows:

1. Interchange each occurrence of "source" in σ with "target".
2. Interchange the order of composing morphisms. That is, replace each occurrence of $g \circ f$ with $f \circ g$

Informally, these conditions state that the dual of a statement is formed by reversing arrows and compositions.

Duality is the observation that σ is true for some category C if and only if σ^{op} is true for C^{op}.

Examples

• A morphism $f\colon A \to B$ is a monomorphism if $f \circ g = f \circ h$ implies $g = h$. Performing the dual operation, we get the statement that $g \circ f = h \circ f$ implies $g = h.$ for a morphism $f\colon B \to A$. This is precisely what it means for f to be an epimorphism. In short, the property of being a monomorphism is dual to the property of being an epimorphism.

Applying duality, this means that a morphism in some category C is a monomorphism if and only if the reverse morphism in the opposite category C^{op} is an epimorphism.

• An example comes from reversing the direction of inequalities in a partial order. So if X is a set and ≤ a partial order relation, we can define a new partial order relation $\leq_{new}$ by

$x \leq_{new} y$ if and only if $y \leq x$.

This example on orders is a special case, since partial orders correspond to a certain kind of category in which Hom(A,B) can have at most one element. In applications to logic, this then looks like a very general description of negation (that is, proofs run in the opposite direction). For example, if we take the opposite of a lattice, we will find that *meets* and *joins* have their roles interchanged. This is an abstract form of De Morgan's laws, or of duality applied to lattices.

- Limits and colimits are dual notions.
- Fibrations and cofibrations are examples of dual notions in algebraic topology and homotopy theory. In this context, the duality is often called Eckmann–Hilton duality.

Abelian category

In mathematics, an **abelian category** is a category in which morphisms and objects can be added and in which kernels and cokernels exist and have desirable properties. The motivating prototype example of an abelian category is the category of abelian groups, **Ab**. The theory originated in a tentative attempt to unify several cohomology theories by Alexander Grothendieck. Abelian categories are very *stable* categories, for example they are regular and they satisfy the snake lemma. The class of Abelian categories is closed under several categorical constructions, for example, the category of chain complexes of an Abelian category, or the category of functors from a small category to an Abelian category are Abelian as well. These stability properties make them inevitable in homological algebra and beyond; the theory has major applications in algebraic geometry, cohomology and pure category theory.

Definitions

A category is **abelian** if

- it has a zero object,
- it has all *pullbacks* and *pushouts*, and
- all monomorphisms and epimorphisms are normal.

By a theorem of Peter Freyd, this definition is equivalent to the following "piecemeal" definition:

- A category is *preadditive* if it is enriched over the monoidal category **Ab** of abelian groups. This means that all hom-sets are abelian groups and the composition of morphisms is bilinear.
- A preadditive category is *additive* if every finite set of objects has a biproduct. This means that we can form finite direct sums and direct products.
- An additive category is *preabelian* if every morphism has both a kernel and a cokernel.
- Finally, a preabelian category is **abelian** if every monomorphism and every epimorphism is normal. This means that every monomorphism is a kernel of some morphism, and every epimorphism is a cokernel of some morphism.

Note that the enriched structure on hom-sets is a *consequence* of the three axioms of the first definition. This highlights the foundational relevance of the category of Abelian groups in the theory and its canonical nature.

The concept of exact sequence arises naturally in this setting, and it turns out that exact functors, i.e. the functors preserving exact sequences in various senses, are the relevant functors between Abelian categories. This *exactness* concept has been axiomatized in the theory of exact categories, forming a very special case of regular categories.

Examples

- As mentioned above, the category of all abelian groups is an abelian category. The category of all finitely generated abelian groups is also an abelian category, as is the category of all finite abelian groups.
- If *R* is a ring, then the category of all left (or right) modules over *R* is an abelian category. In fact, it can be shown that any small abelian category is equivalent to a full subcategory of such a category of modules (*Mitchell's embedding theorem*).
- If *R* is a left-noetherian ring, then the category of finitely generated left modules over *R* is abelian. In particular, the category of finitely generated modules over a noetherian commutative ring is abelian; in this way, abelian categories show up in commutative algebra.
- As special cases of the two previous examples: the category of vector spaces over a fixed field *k* is abelian, as is the category of finite-dimensional vector spaces over *k*.
- If *X* is a topological space, then the category of all (real or complex) vector bundles on *X* is not usually an abelian category, as there can be monomorphisms that are not kernels.
- If *X* is a topological space, then the category of all sheaves of abelian groups on *X* is an abelian category. More generally, the category of sheaves of abelian groups on a Grothendieck site is an abelian category. In this way, abelian categories show up in algebraic topology and algebraic geometry.
- If **C** is a small category and **A** is an abelian category, then the category of all functors from **C** to **A** forms an abelian category (the morphisms of this category are the natural transformations between functors). If **C** is small and preadditive, then the category of all additive functors from **C** to **A** also forms an abelian category. The latter is a generalization of the *R*-module example, since a ring can be understood as a preadditive category with a single object.

Grothendieck's axioms

In his Tôhoku article, Grothendieck listed four additional axioms (and their duals) that an abelian category **A** might satisfy. These axioms are still in common use to this day. They are the following:

- AB3) For every set $\{A_i\}$ of objects of **A**, the coproduct $\coprod A_i$ exists in **A** (i.e. **A** is cocomplete).
- AB4) **A** satisfies AB3), and the coproduct of a family of monomorphisms is a monomorphism.
- AB5) **A** satisfies AB3), and filtered colimits of exact sequences are exact.

and their duals

- AB3*) For every set $\{A_i\}$ of objects of **A**, the product ΠA_i exists in **A** (i.e. **A** is complete).
- AB4*) **A** satisfies AB3*), and the product of a family of epimorphisms is an epimorphism.
- AB5*) **A** satisfies AB3*), and filtered limits of exact sequences are exact.

Axioms AB1) and AB2) were also given. They are what make an additive category abelian. Specifically:

- AB1) Every morphism has a kernel and a cokernel.
- AB2) For every morphism *f*, the canonical morphism from coim *f* to im *f* is an isomorphism.

Grothendieck also gave axioms AB6) and AB6*).

Elementary properties

Given any pair A, B of objects in an abelian category, there is a special zero morphism from A to B. This can be defined as the zero element of the hom-set Hom(A,B), since this is an abelian group. Alternatively, it can be defined as the unique composition $A \to 0 \to B$, where 0 is the zero object of the abelian category.

In an abelian category, every morphism f can be written as the composition of an epimorphism followed by a monomorphism. This epimorphism is called the *coimage* of f, while the monomorphism is called the *image* of f.

Subobjects and quotient objects are well-behaved in abelian categories. For example, the poset of subobjects of any given object A is a bounded lattice.

Every abelian category **A** is a module over the monoidal category of finitely generated abelian groups; that is, we can form a tensor product of a finitely generated abelian group G and any object A of **A**. The abelian category is also a comodule; Hom(G,A) can be interpreted as an object of **A**. If **A** is complete, then we can remove the requirement that G be finitely generated; most generally, we can form finitary enriched limits in **A**.

Related concepts

Abelian categories are the most general setting for homological algebra. All of the constructions used in that field are relevant, such as exact sequences, and especially short exact sequences, and derived functors. Important theorems that apply in all abelian categories include the five lemma (and the short five lemma as a special case), as well as the snake lemma (and the nine lemma as a special case).

History

Abelian categories were introduced by Buchsbaum (1955) (under the name of "exact category") and Grothendieck (1957) in order to unify various cohomology theories. At the time, there was a cohomology theory for sheaves, and a cohomology theory for groups. The two were defined differently, but they had similar properties. In fact, much of category theory was developed as a language to study these similarities. Grothendieck unified the two theories: they both arise as derived functors on abelian categories; the abelian category of sheaves of abelian groups on a topological space, and the abelian category of G-modules for a given group G.

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Yoneda lemma

In mathematics, specifically in category theory, the **Yoneda lemma** is an abstract result on functors of the type *morphisms into a fixed object*. It is a vast generalisation of Cayley's theorem from group theory (viewing a group as a particular kind of category with just one object). It allows the embedding of any category into a category of functors defined on that category. It also clarifies how the embedded category, of representable functors and their natural transformations, relates to the other objects in the larger functor category. It is an important tool that underlies several modern developments in algebraic geometry and representation theory. It is named after Nobuo Yoneda.

Generalities

The Yoneda lemma suggests that instead of studying the (small) category *C*, one should study the category of all functors of *C* into **Set** (the category of sets with functions as morphisms). **Set** is a category we understand well, and a functor of *C* into **Set** can be seen as a "representation" of *C* in terms of known structures. The original category *C* is contained in this functor category, but new objects appear in the functor category which were absent and "hidden" in *C*. Treating these new objects just like the old ones often unifies and simplifies the theory.

This approach is akin to (and in fact generalizes) the common method of studying a ring by investigating the modules over that ring. The ring takes the place of the category *C*, and the category of modules over the ring is a category of functors defined on *C*.

Formal statement

General version

Yoneda's lemma concerns functors from a fixed category *C* to the category of sets, **Set**. If *C* is a locally small category (i.e. the hom-sets are actual sets and not proper classes), then each object *A* of *C* gives rise to a natural functor to **Set** called a hom-functor. This functor is denoted:

$$h^A = \mathrm{Hom}(A, -),$$

The hom-functor h^A sends *X* to the set of morphisms Hom(*A*,*X*) and sends a morphism *f* from *X* to *Y* to the morphism $f \circ -$ (composition with *f* on the left) that sends a morphism *g* in Hom(*A*,*X*) to the morphism $f \circ g$ in Hom(*A*,*Y*).

Let *F* be an arbitrary functor from *C* to **Set**. Then Yoneda's lemma says that for each object *A* of *C*, the natural transformations from h^A to *F* are in one-to-one correspondence with the elements of *F*(*A*). That is,

$$\mathrm{Nat}(h^A, F) \cong F(A).$$

Given a natural transformation Φ from h^A to *F*, the corresponding element of *F*(*A*) is $u = \Phi_A(\mathrm{id}_A)$.

There is a contravariant version of Yoneda's lemma which concerns contravariant functors from *C* to **Set**. This version involves the contravariant hom-functor

$$h_A = \mathrm{Hom}(-, A),$$

which sends *X* to the hom-set Hom(*X*,*A*). Given an arbitrary contravariant functor *G* from *C* to **Set**, Yoneda's lemma asserts that

$$\mathrm{Nat}(h_A, G) \cong G(A).$$

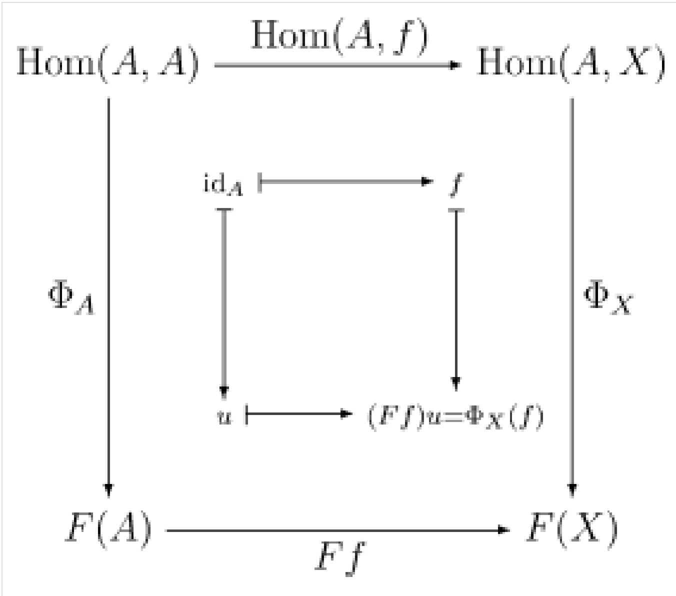
Naming conventions

The use of "h^A" for the covariant hom-functor and "h_A" for the contravariant hom-functor is not completely standard. However, the exceptions to this rule are usually completely unrelated symbols, and it is very rare to see "h_A" used to mean the covariant hom-functor or vice-versa.[1]

The mnemonic "falling into something" can be helpful in remembering that "h_A" is the contravariant hom-functor. When the letter "A" is **falling** (i.e. a subscript), h_A assigns to an object X the morphisms from X **into** A.

Proof

The proof of Yoneda's lemma is indicated by the following commutative diagram:



This diagram shows that the natural transformation Φ is completely determined by $\Phi_A(\mathrm{id}_A) = u$ since for each morphism $f : A \to X$ one has

$$\Phi_X(f) = (Ff)u.$$

Moreover, any element $u \in F(A)$ defines a natural transformation in this way. The proof in the contravariant case is completely analogous.

In this way, Yoneda's Lemma provides a complete classification of all natural transformations from the functor Hom(A,-) to an arbitrary functor F:C→Set.

The Yoneda embedding

An important special case of Yoneda's lemma is when the functor F from C to **Set** is another hom-functor h^B. In this case, the covariant version of Yoneda's lemma states that

$$\mathrm{Nat}(h^A, h^B) \cong \mathrm{Hom}(B, A).$$

That is, natural transformations between hom-functors are in one-to-one correspondence with morphisms (in the reverse direction) between the associated objects. Given a morphism $f : B \to A$ the associated natural transformation is denoted Hom(f,–).

Mapping each object A in C to its associated hom-functor h^A = Hom(A,–) and each morphism $f : B \to A$ to the corresponding natural transformation Hom(f,–) determines a contravariant functor h^- from C to $\mathbf{Set}^C$, the functor category of all (covariant) functors from C to **Set**. One can interpret h^- as a covariant functor:

$$h^- : \mathcal{C}^{\mathrm{op}} \to \mathbf{Set}^{\mathcal{C}}.$$

The meaning of Yoneda's lemma in this setting is that the functor h^{-} is fully faithful, and therefore gives an embedding of C^{op} in the category of functors to **Set**. The collection of all functors $\{h^{A}$, A in C$\}$ is a subcategory of $\mathbf{Set}^{C}$. Therefore, Yoneda embedding implies that the category $\mathbf{C}^{op}$ is isomorphic to the category $\{h^{A}$, A in C$\}$.

The contravariant version of Yoneda's lemma states that

$$\mathrm{Nat}(h_A, h_B) \cong \mathrm{Hom}(A, B).$$

Therefore, $h_{\_}$ gives rise to a covariant functor from C to the category of contravariant functors to **Set**:

$$h_{-} : \mathcal{C} \to \mathbf{Set}^{\mathcal{C}^{\mathrm{op}}}.$$

Yoneda's lemma then states that any locally small category C can be embedded in the category of contravariant functors from C to **Set** via $h_{\_}$. This is called the *Yoneda embedding*.

Preadditive categories, rings and modules

A *preadditive category* is a category where the morphism sets form abelian groups and the composition of morphisms is bilinear; examples are categories of abelian groups or modules. In a preadditive category, there is both a "multiplication" and an "addition" of morphisms, which is why preadditive categories are viewed as generalizations of rings. Rings are preadditive categories with one object.

The Yoneda lemma remains true for preadditive categories if we choose as our extension the category of *additive* contravariant functors from the original category into the category of abelian groups; these are functors which are compatible with the addition of morphisms and should be thought of as forming a *module category* over the original category. The Yoneda lemma then yields the natural procedure to enlarge a preadditive category so that the enlarged version remains preadditive — in fact, the enlarged version is an abelian category, a much more powerful condition. In the case of a ring R, the extended category is the category of all left modules over R, and the statement of the Yoneda lemma reduces to the well-known isomorphism

$M \cong \mathrm{Hom}_R(R,M)$ for all left modules M over R.

Notes

[1] A notable exception is *Commutative algebra with a view toward algebraic geometry* / David Eisenbud (1995), which uses "h_A" to mean the covariant hom-functor. However, the later book *The geometry of schemes* / David Eisenbud, Joe Harris (1998) reverses this and uses "h_A" to mean the contravariant hom-functor.

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Limit (category theory)

In category theory, a branch of mathematics, the abstract notion of a **limit** captures the essential properties of universal constructions such as products and inverse limits. The dual notion of a **colimit** generalizes constructions such as disjoint unions, direct sums, coproducts, pushouts and direct limits.

Limits and colimits, like the strongly related notions of universal properties and adjoint functors, exist at a high level of abstraction. In order to understand them, it is helpful to first study the specific examples these concepts are meant to generalize.

Definition

Limits and colimits in a category C are defined by means of diagrams in C. Formally, a **diagram** of type J in C is a functor from J to C:

$$F : J \to C.$$

The category J is thought of as index category, and the diagram F is thought of as indexing a collection of objects and morphisms in C patterned on J. The actual objects and morphisms in J are largely irrelevant—only the way in which they are interrelated matters.

One is most often interested in the case where the category J is a small or even finite category. A diagram is said to be **small** or **finite** whenever J is.

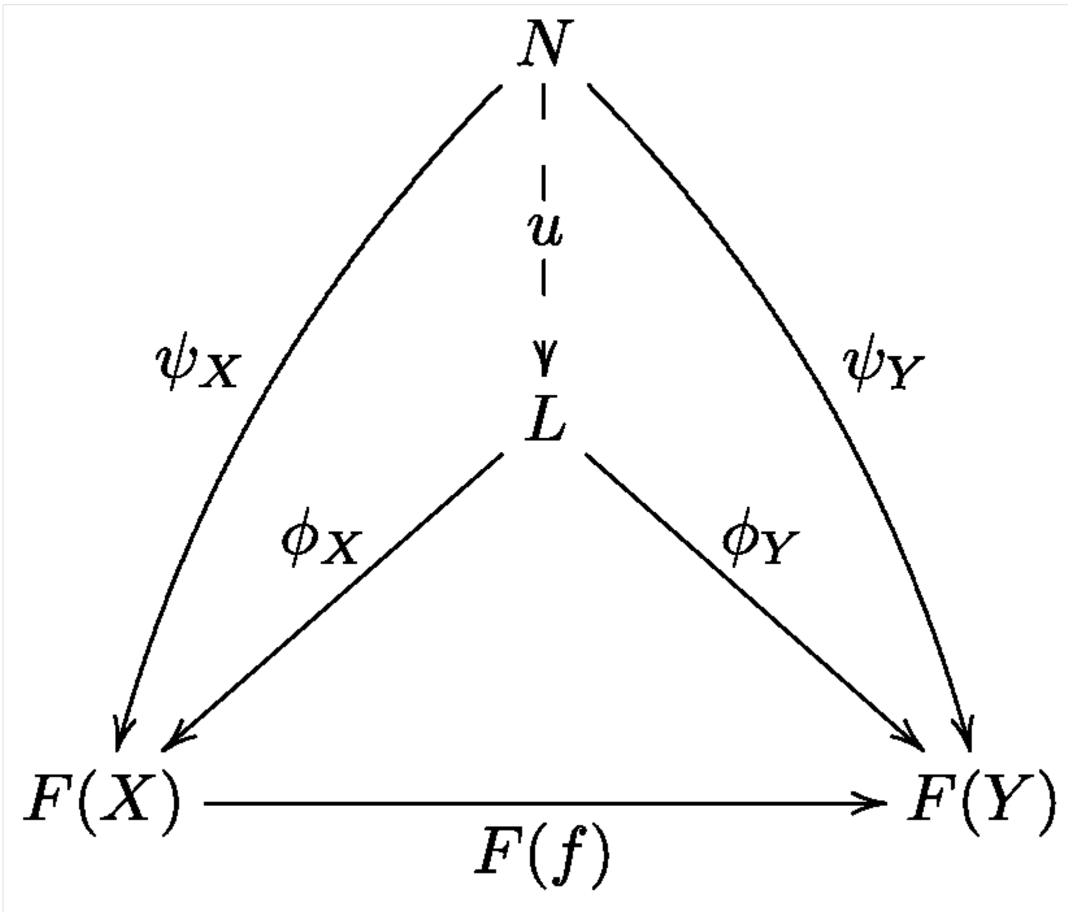
Limits

Let $F : J \to C$ be a diagram of type J in a category C. A **cone** to F is an object N of C together with a family of morphisms

$$\psi_X : N \to F(X),$$

one for each object X of J, such that for every morphism $f : X \to Y$ in J, we have $F(f) \circ \psi_X = \psi_Y$.

A **limit** of the diagram $F : J \to C$ is a cone (L, φ) to F such that for any other cone (N, ψ) to F there exists a *unique* morphism $u : N \to L$ such that $\varphi_X \circ u = \psi_X$ for all X in J.



One says that the cone (N, ψ) factors through the cone (L, φ) with the unique factorization u. The morphism u is sometimes called the **mediating morphism**.

Limits are also referred to as *universal cones*, since they are characterized by a universal property (see below for more information). As with every universal property, the above definition describes a balanced state of generality: The limit object L has to be general enough to allow any other cone to factor through it; on the other hand, L has to be sufficiently specific, so that only *one* such factorization is possible for every cone.

Limits may also be characterized as terminal objects in the category of cones to F.

It is possible that a diagram does not have a limit at all. However, if a diagram does have a limit then this limit is essentially unique: it is unique up to a unique isomorphism. For this reason one often speaks of *the* limit of F.

Colimits

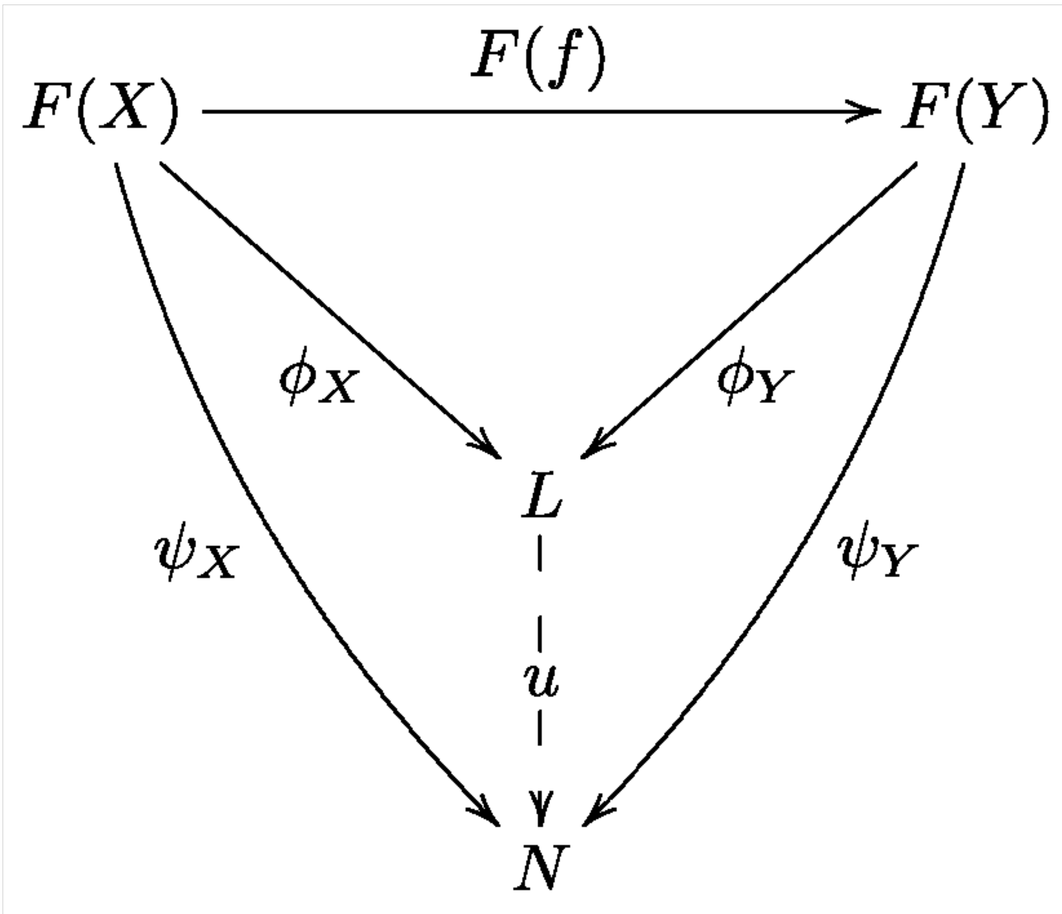
The dual notions of limits and cones are colimits and co-cones. Although it is straightforward to obtain the definitions of these by inverting all morphisms in the above definitions, we will explicitly state them here:

A **co-cone** of a diagram $F : J \to C$ is an object N of C together with a family of morphisms

$$\psi_X : F(X) \to N$$

for every object X of J, such that for every morphism $f : X \to Y$ in J, we have $\psi_Y \circ F(f) = \psi_X$.

A **colimit** of a diagram $F : J \to C$ is a co-cone (L, ϕ) of F such that for any other co-cone (N, ψ) of F there exists a unique morphism $u : L \to N$ such that $u \circ \phi_X = \psi_X$ for all X in J.



Colimits are also referred to as *universal co-cones*. They can be characterized as initial objects in the category of co-cones from F.

As with limits, if a diagram F has a colimit then this colimit is unique up to a unique isomorphism.

Variations

Limits and colimits can also be defined for collections of objects and morphisms without the use of diagrams. The definitions are the same (note that in definitions above we never needed to use composition of morphisms in J). This variation, however, adds no new information. Any collection of objects and morphisms defines a (possibly large) directed graph G. If we let J be the free category generated by G, there is a universal diagram $F : J \to C$ whose image contains G. The limit (or colimit) of this diagram is the same as the limit (or colimit) of the original collection of objects and morphisms.

Weak limit and **weak colimits** are defined like limits and colimits, except that the uniqueness property of the mediating morphism is dropped.

Examples

Limits

The definition of limits is general enough to subsume several constructions useful in practical settings. In the following we will consider the limit (L, φ) of a diagram $F : J \to C$.

- **Terminal objects**. If J is the empty category there is only one diagram of type J: the empty one (similar to the empty function in set theory). A cone to the empty diagram is essentially just an object of C. The limit of F is any object that has a unique factorization through any other object. This is just the definition of a *terminal object*.
- **Products**. If J is a discrete category then a diagram F is essentially nothing but a family of objects of C, indexed by J. The limit L of F is called the *product* of these objects. The cone φ consists of a family of morphisms $\varphi_X : L \to F(X)$ called the *projections* of the product. In the category of sets, for instance, the products are given by Cartesian products and the projections are just the natural projections onto the various factors.
  - **Powers**. A special case of a product is when the diagram F is a constant functor to an object X of C. The limit of this diagram is called the J^{th} *power* of X and denoted X^J.
- **Equalizers**. If J is a category with two objects and two parallel morphisms from object *1* to object *2* then a diagram of type J is a pair of parallel morphisms in C. The limit L of such a diagram is called an *equalizer* of those morphisms.
  - **Kernels**. A *kernel* is a special case of an equalizer where one of the morphisms is a zero morphism.
- **Pullbacks**. Let F be a diagram that picks out three objects X, Y, and Z in C, where the only non-identity morphisms are $f : X \to Z$ and $g : Y \to Z$. The limit L of F is called a *pullback* or a *fiber product*. It can nicely be visualized as a commutative square:

$$\begin{tikzcd} L \arrow[r, "\phi_X"] \arrow[d, "\phi_Y"'] \arrow[dr, "\phi_Z"] & X \arrow[d, "f"] \\ Y \arrow[r, "g"'] & Z \end{tikzcd}$$

- **Inverse limits**. Let J be a directed poset (considered as a small category by adding arrows $i \to j$ if and only if $i \leq j$) and let $F : J^{\mathrm{op}} \to C$ be a diagram. The limit of F is called (confusingly) an *inverse limit*, *projective limit*, or *directed limit*.
- If $J = \mathbf{1}$, the category with a single object and morphism, then a diagram of type J is essentially just an object of C. A cone to an object X is just a morphism with codomain X. A morphism $f : Y \to X$ is a limit of the diagram X if and only if f is an isomorphism. More generally, if J is any category with an initial object i, then any diagram of type J has a limit, namely any object isomorphic to $F(i)$. Such an isomorphism uniquely determines a universal cone to F.

Colimits

Examples of colimits are given by the dual versions of the examples above:

- **Initial objects** are colimits of empty diagrams.
- **Coproducts** are colimits of diagrams indexed by discrete categories.
  - **Copowers** are colimits of constant diagrams from discrete categories.
- **Coequalizers** are colimits of a parallel pair of morphisms.
  - **Cokernels** are coequalizers of a morphism and a parallel zero morphism.
- **Pushouts** are colimits of a pair of morphisms with common domain.
- **Direct limits** are colimits of diagrams indexed by directed sets.

Properties

Existence of limits

A given diagram $F : J \to C$ may or may not have a limit (or colimit) in C. Indeed, there may not even be a cone to F, let alone a universal cone.

A category C is said to **have limits of type *J*** if every diagram of type J has a limit in C. Specifically, a category C is said to

- **have products** if it has limits of type J for every *small* discrete category J (it need not have large products),
- **have equalizers** if it has limits of type $\bullet \rightrightarrows \bullet$ (i.e. every parallel pair of morphisms has an equalizer),
- **have pullbacks** if it has limits of type $\bullet \rightarrow \bullet \leftarrow \bullet$ (i.e. every pair of morphisms with common codomain has a pullback).

A **complete category** is a category that has all small limits (i.e. all limits of type J for every small category J).

One can also make the dual definitions. A category **has colimits of type *J*** if every diagram of type J has a colimit in C. A **cocomplete category** is one that has all small colimits.

The **existence theorem for limits** states that if a category C has equalizers and all products indexed by the classes Ob(J) and Hom(J), then C has all limits of type J. In this case, the limit of a diagram $F : J \to C$ can be constructed as the equalizer of the two morphisms

$$s,t : \prod_{i\in \mathrm{Ob}(J)} F_i \rightrightarrows \prod_{f\in \mathrm{Hom}(J)} F_{\mathrm{cod}(f)}$$

given (in component form) by

$$s = \big\langle F(f) \circ \pi_{\mathrm{dom}(f)} \big\rangle$$
$$t = \big\langle \pi_{\mathrm{cod}(f)} \big\rangle.$$

There is a dual **existence theorem for colimits** in terms of coequalizers and coproducts. Both of these theorems give sufficient but not necessary conditions for the existence of all (co)limits of type J.

Universal property

Limits and colimits are important special cases of universal constructions. Let C be a category and let J be a small index category. The functor category C^J may be thought of the category of all diagrams of type J in C. The *diagonal functor*

$$\Delta : \mathcal{C} \to \mathcal{C}^{\mathcal{J}}$$

is the functor that maps each object N in C to the constant functor $\Delta(N) : J \to C$ to N. That is, $\Delta(N)(X) = N$ for each object X in J and $\Delta(N)(f) = \mathrm{id}_N$ for each morphism f in J.

Given a diagram F: $J \to C$ (thought of as an object in C^J), a natural transformation $\psi : \Delta(N) \to F$ (which is just a morphism in the category C^J) is the same thing as a cone from N to F. The components of ψ are the morphisms ψ_X : $N \to F(X)$. Dually, a natural transformation $\psi : F \to \Delta(N)$ is the same thing as a co-cone from F to N.

The definitions of limits and colimits can then be restated in the form:

- A limit of F is a universal morphism from Δ to F.
- A colimit of F is a universal morphism from F to Δ.

Adjunctions

Like all universal constructions, the formation of limits and colimits is functorial in nature. In other words, if every diagram of type J has a limit in C (for J small) there exists a **limit functor**

$$\lim : \mathcal{C}^{\mathcal{J}} \to \mathcal{C}$$

which assigns each diagram its limit and each natural transformation $\eta : F \to G$ the unique morphism $\lim \eta : \lim F \to \lim G$ commuting with the corresponding universal cones. This functor is right adjoint to the diagonal functor $\Delta : C \to C^J$. This adjunction gives a bijection between the set of all morphisms from N to $\lim F$ and the set of all cones from N to F

$$\mathrm{Hom}(N, \mathrm{lim}F) \cong \mathrm{Cone}(N, F)$$

which is natural in the variables N and F. The counit of this adjunction is simply the universal cone from $\lim F$ to F. If the index category J is connected (and nonempty) then the unit of the adjunction is an isomorphism so that lim is a left inverse of Δ. This fails if J is not connected. For example, if J is a discrete category, the components of the unit are the diagonal morphisms $\delta : N \to N^J$.

Dually, if every diagram of type J has a colimit in C (for J small) there exists a **colimit functor**

$$\mathrm{colim} : \mathcal{C}^{\mathcal{J}} \to \mathcal{C}$$

which assigns each diagram its colimit. This functor is left adjoint to the diagonal functor $\Delta : C \to C^J$, and one has a natural isomorphism

$$\mathrm{Hom}(\mathrm{colim}F, N) \cong \mathrm{Cocone}(F, N).$$

The unit of this adjunction is the universal cocone from F to $\mathrm{colim}\ F$. If J is connected (and nonempty) then the counit is an isomorphism, so that colim is a left inverse of Δ.

Note that both the limit and the colimit functors are *covariant* functors.

As representations of functors

One can use Hom functors to relate limits and colimits in a category C to limits in **Set**, the category of sets. This follows, in part, from the fact the covariant Hom functor $\mathrm{Hom}(N, -) : C \to \mathbf{Set}$ preserves all limits in C. By duality, the contravariant Hom functor must take colimits to limits.

If a diagram $F : J \to C$ has a limit in C, denoted by $\lim F$, there is a canonical isomorphism

$$\mathrm{Hom}(N, \lim F) \cong \lim \mathrm{Hom}(N, F-)$$

which is natural in the variable N. Here the functor $\mathrm{Hom}(N, F-)$ is the composition of the Hom functor $\mathrm{Hom}(N, -)$ with F. This isomorphism is the unique one which respects the limiting cones.

One can use the above relationship to define the limit of F in C. The first step is to observe that the limit of the functor $\mathrm{Hom}(N, F-)$ can be identified with the set of all cones from N to F:

$$\lim \mathrm{Hom}(N, F-) = \mathrm{Cone}(N, F).$$

The limiting cone is given by the family of maps $\pi_X : \mathrm{Cone}(N, F) \to \mathrm{Hom}(N, FX)$ where $\pi_X(\psi) = \psi_X$. If one is given an object L of C together with a natural isomorphism $\Phi : \mathrm{Hom}(-, L) \to \mathrm{Cone}(-, F)$, the object L will be a limit of F with the limiting cone given by $\Phi_L(\mathrm{id}_L)$. In fancy language, this amounts to saying that a limit of F is a representation of the functor $\mathrm{Cone}(-, F) : C \to \mathbf{Set}$.

Dually, if a diagram $F : J \to C$ has a colimit in C, denoted colim F, there is a unique canonical isomorphism

$$\mathrm{Hom}(\mathrm{colim} F, N) \cong \lim \mathrm{Hom}(F-, N)$$

which is natural in the variable N and respects the colimiting cones. Identifying the limit of $\mathrm{Hom}(F-, N)$ with the set $\mathrm{Cocone}(F, N)$, this relationship can be used to define the colimit of the diagram F as a representation of the functor $\mathrm{Cocone}(F, -)$.

Interchange of limits and colimits of sets

Let I be a finite category and J be a small filtered category. For any bifunctor

$$F : I \times J \to \mathbf{Set}$$

there is a natural isomorphism

$$\mathrm{colim}_J \lim_I F(i, j) \to \lim_I \mathrm{colim}_J F(i, j).$$

In words, filtered colimits in **Set** commute with finite limits.

Functors and limits

If $F : J \to C$ is a diagram in C and $G : C \to D$ is a functor then by composition (recall that a diagram is just a functor) one obtains a diagram $GF : J \to D$. A natural question is then:

"How are the limits of GF related to those of F?"

Preservation of limits

A functor $G : C \to D$ induces a map from Cone(F) to Cone(GF): if Ψ is a cone from N to F then $G\Psi$ is a cone from GN to GF. The functor G is said to **preserve the limits of *F*** if (GL, $G\varphi$) is a limit of GF whenever (L, φ) is a limit of F. (Note that if the limit of F does not exist, then G vacuously preserves the limits of F.)

A functor G is said to **preserve all limits of type *J*** if it preserves the limits of all diagrams $F : J \to C$. For example, one can say that G preserves products, equalizers, pullbacks, etc. A **continuous functor** is one that preserves all *small* limits.

One can make analogous definitions for colimits. For instance, a functor G preserves the colimits of F if $G(L, \varphi)$ is a colimit of GF whenever (L, φ) is a colimit of F. A **cocontinuous functor** is one that preserves all *small* colimits.

If C is a complete category, then, by the above existence theorem for limits, a functor $G : C \to D$ is continuous if and only if it preserves (small) products and equalizers. Dually, G is cocontinuous if and only if it preserves (small) coproducts and coequalizers.

An important property of adjoint functors is that every right adjoint functor is continuous and every left adjoint functor is cocontinuous. Since adjoint functors exist in abundance, this gives numerous examples of continuous and cocontinuous functors.

For a given diagram $F : J \to C$ and functor $G : C \to D$, if both F and GF have specified limits there is a unique canonical morphism

$$\tau_F : G \lim F \to \lim GF$$

which respects the corresponding limit cones. The functor G preserves the limits of F if and only this map is an isomorphism. If the categories C and D have all limits of type J then lim is a functor and the morphisms τ_F form the components of a natural transformation

$$\tau : G \lim \to \lim G^J.$$

The functor G preserves all limits of type J if and only if τ is a natural isomorphism. In this sense, the functor G can be said to *commute with limits* (up to a canonical natural isomorphism).

Preservation of limits and colimits is a concept that only applies to *covariant* functors. For contravariant functors the corresponding notions would be a functor that takes colimits to limits, or one that takes limits to colimits.

Lifting of limits

A functor $G : C \to D$ is said to **lift limits** for a diagram $F : J \to C$ if whenever (L, φ) is a limit of GF there exists a limit (L', φ') of F such that $G(L', \varphi') = (L, \varphi)$. A functor G **lifts limits of type *J*** if it lifts limits for all diagrams of type J. One can therefore talk about lifting products, equalizers, pullbacks, etc. Finally, one says that G **lifts limits** if it lifts all limits. There are dual definitions for the lifting of colimits.

A functor G **lifts limits uniquely** for a diagram F if the preimage cone (L', φ') is the unique limit of F such that $G(L', \varphi') = (L, \varphi)$. One can show that G lifts limits uniquely if and only if it lifts limits and is amnestic.

Lifting of limits is clearly related to preservation of limits. If G lifts limits for a diagram F and GF has a limit, then F also has a limit and G preserves the limits of F. It follows that:

- If G lifts limits of all type J and D has all limits of type J, then C also has all limits of type J and G preserves these limits.
- If G lifts all small limits and D is complete, then C is also complete and G is continuous.

The dual statements for colimits are equally valid.

Creation and reflection of limits

Let $F : J \to C$ be a diagram. A functor $G : C \to D$ is said to

- **create limits** for F if whenever (L, φ) is a limit of GF there exists a unique cone (L', φ') to F such that $G(L', \varphi') = (L, \varphi)$, and furthermore, this cone is a limit of F.
- **reflect limits** for F if each cone to F whose image under G is a limit of GF is already a limit of F.

Dually, one can define creation and reflection of colimits.

The following statements are easily seen to be equivalent:

- The functor G creates limits.
- The functor G lifts limits uniquely and reflects limits.

There are examples of functors which lift limits uniquely but neither create nor reflect them.

Examples

- For any category C and object A of C the Hom functor Hom(A,–) : $C \to$ **Set** preserves all limits in C. In particular, Hom functors are continuous. Hom functors need not preserve colimits.
- Every representable functor $C \to$ **Set** preserves limits (but not necessarily colimits).
- The forgetful functor U : **Grp** → **Set** creates (and preserves) all small limits and filtered colimits; however, U does not preserve coproducts. This situation is typical of algebraic forgetful functors.
- The free functor F : **Set** → **Grp** (which assigns to every set S the free group over S) is left adjoint to forgetful functor U and is, therefore, cocontinuous. This explains why the free product of two free groups G and H is the free group generated by the disjoint union of the generators of G and H.
- The inclusion functor **Ab** → **Grp** creates limits but does not preserve coproducts (the coproduct of two abelian groups being the direct sum).
- The forgetful functor **Top** → **Set** lifts limits and colimits uniquely but creates neither.
- Let $\mathbf{Met}_c$ be the category of metric spaces with continuous functions for morphisms. The forgetful functor $\mathbf{Met}_c \to$ **Set** lifts finite limits but does not lift them uniquely.

A note on terminology

Older terminology referred to limits as "inverse limits" or "projective limits," and to colimits as "direct limits" or "inductive limits." This has been the source of a lot of confusion.

There are several ways to remember the modern terminology. First of all,

- cokernels,
- coequalizers, and
- codomains

are types of colimits, whereas

- kernels,
- equalizers, and
- domains

are types of limits. Second, the prefix "co" implies "first variable of the Hom". Terms like "cohomology" and "cofibration" all have a slightly stronger association with the first variable, i.e., the contravariant variable, of the Hom bifunctor.

References

- Adámek, Jiří; Horst Herrlich, and George E. Strecker (1990). *Abstract and Concrete Categories* [1]. John Wiley & Sons. ISBN 0-471-60922-6.
- Mac Lane, Saunders (1998). *Categories for the Working Mathematician*. Graduate Texts in Mathematics **5** (2nd ed. ed.). Springer. ISBN 0-387-98403-8.

External links

- Interactive Web page [2] which generates examples of limits and colimits in the category of finite sets. Written by Jocelyn Paine [3].

References

[1] http://katmat.math.uni-bremen.de/acc/acc.pdf
[2] http://www.j-paine.org/cgi-bin/webcats/webcats.php
[3] http://www.j-paine.org/

Adjoint functors

In mathematics, **adjoint functors** are pairs of functors which stand in a particular relationship with one another, called an **adjunction**. The relationship of adjunction is ubiquitous in mathematics, as it rigorously reflects the intuitive notions of optimization and efficiency. It is studied in generality by the branch of mathematics known as category theory, which helps to minimize the repetition of the same logical details separately in every subject.

In the most concise symmetric definition, an adjunction between categories *C* and *D* is a pair of functors,

$$F : \mathcal{D} \to \mathcal{C} \quad \text{and} \quad G : \mathcal{C} \to \mathcal{D}$$

and a family of bijections

$$\mathrm{hom}_{\mathcal{C}}(FY, X) \cong \mathrm{hom}_{\mathcal{D}}(Y, GX)$$

which is natural in the variables *X* and *Y*. The functor *F* is called a **left adjoint functor**, while *G* is called a **right adjoint functor**. The relationship "*F* is left adjoint to *G*" (or equivalently, "*G* is right adjoint to *F*") is sometimes written

$$F \dashv G.$$

This definition and others are made precise below.

Introduction

"The slogan is 'Adjoint functors arise everywhere'." (Saunders Mac Lane, *Categories for the working mathematician*)

The long list of examples in this article is only a partial indication of how often an interesting mathematical construction is an adjoint functor. As a result, general theorems about left/right adjoint functors, such as the equivalence of their various definitions or the fact that they respectively preserve colimits/limits (which are also found in every area of math), can encode the details of many useful and otherwise non-trivial results.

Motivation

One good way to motivate adjoint functors is to vaguely explain what problem they solve, and how they solve it. (This motivation runs parallel to the definitions via universal morphisms below.)

Adjoint functors as formulaic solutions to optimization problems

It can be said that an adjoint functor is a way of giving the *most efficient* solution to some problem via a method which is *formulaic*. For example, an elementary problem in ring theory is how to turn a rng (which is like a ring that might not have a multiplicative identity) into a ring. The *most efficient* way is to adjoin an element '1' to the rng, adjoin no unnecessary extra elements (we will need to have *r*+1 for each *r* in the ring, clearly), and impose no relations in the newly formed ring that are not forced by axioms. Moreover, this construction is *formulaic* in the sense that it works in essentially the same way for any rng.

This is rather vague, though suggestive, and can be made precise in the language of category theory: a construction is *most efficient* if it satisfies a universal property, and is *formulaic* if it defines a functor. Universal properties come in two types: initial properties and terminal properties. Since these are dual (opposite) notions, it is only necessary to discuss one of them.

The idea of using an initial property is to set up the problem in terms of some auxiliary category E, and then identify that what we want is to find an initial object of E. This has an advantage that the *optimization* — the sense that we are finding the *most efficient* solution — means something rigorous and is recognisable, rather like the attainment of a supremum. Picking the right category E is something of a knack: for example, take the given rng R, and make a category E whose *objects* are rng homomorphisms $R \to S$, with S a ring having a multiplicative identity. The *morphisms* in E are commutative triangles of the form $(R \to S_1, R \to S_2, S_1 \to S_2)$ where $\mathrm{S}_1 \to \mathrm{S}_2$ is a ring map (which preserves the identity). The assertion that an object $R \to R^*$ is initial in E means that the ring R^* is a *most efficient* solution to our problem.

The two facts that this method of turning rngs into rings is *most efficient* and *formulaic* can be expressed simultaneously by saying that it defines an *adjoint functor*.

The hidden symmetry of optimization problems

Continuing this discussion, suppose we *started* with the functor F, and posed the following (vague) question: is there a problem to which F is the most efficient solution?

The notion that F is the *most efficient solution* to the problem posed by G is, in a certain rigorous sense, equivalent to the notion that G poses the *most difficult problem* which F solves.

This has the intuitive meaning that adjoint functors should occur in pairs, and in fact they do, but this is not trivial from the universal morphism definitions. The equivalent symmetric definitions involving adjunctions and the symmetric language of adjoint functors (we can say either F is left adjoint to G or G is right adjoint to F) have the advantage of making this fact explicit.

Formal definitions

There are various definitions for adjoint functors. Their equivalence is elementary but not at all trivial and in fact highly useful. This article provides several such definitions:

- The definitions via universal morphisms are easy to state, and require minimal verifications when constructing an adjoint functor or proving two functors are adjoint. They are also the most analogous to our intuition involving optimizations.
- The definition via counit-unit adjunction is convenient for proofs about functors which are known to be adjoint, because they provide formulas that can be directly manipulated.
- The definition via hom-sets makes symmetry the most apparent, and is the reason for using the word *adjoint*.

Adjoint functors arise everywhere, in all areas of mathematics. Their full usefulness lies in that the structure in any of these definitions gives rise to the structures in the others via a long but trivial series of deductions. Thus, switching between them makes implicit use of a great deal of tedious details that would otherwise have to be repeated separately in every subject area. For example, naturality and terminality of the counit can be used to prove that any right adjoint functor preserves limits.

A helpful writing convention

The theory of adjoints has the terms *left* and *right* at its foundation, and there are many components which live in one of two categories C and D which are under consideration. It can therefore be extremely helpful to choose letters in alphabetical order according to whether they live in the "lefthand" category C or the "righthand" category D, and also to write them down in this order whenever possible.

In this article for example, the letters X, F, f, ε will consistently denote things which live in the category C, the letters Y, G, g, η will consistently denote things which live in the category D, and whenever possible such things will be referred to in order from left to right (a functor $F:C\leftarrow D$ can be thought of as "living" where its outputs are, in C).

Definitions via universal morphisms

A functor $F : C \leftarrow D$ is a **left adjoint functor** if for each object X in C, there exists a terminal morphism from F to X. If, for each object X in C, we choose an object G_0X of D and a terminal morphism $\varepsilon_X : F(G_0X) \to X$ from F to X, then there is a unique functor $G : C \to D$ such that $GX = G_0X$ and $\varepsilon_{X'} FG(f) = f\,\varepsilon_X$ for $f : X \to X'$ a morphism in C; F is then called a **left adjoint to** G.

A functor $G : C \to D$ is a **right adjoint functor** if for each object Y in D, there exists an initial morphism from Y to G. If, for each object Y in D, we choose an object F_0Y of C and an initial morphism $\eta_Y : Y \to G(F_0Y)$ from Y to G, then there is a unique functor $F : C \leftarrow D$ such that $FY = F_0Y$ and $GF(g)\ \eta_Y = \eta_{Y'}\, g$ for for $g : Y \to Y'$ a morphism in D; G is then called a **right adjoint to** F.

Remarks:

It is true, as the terminology implies, that F is *left adjoint to* G if and only if G is *right adjoint to* F. This is apparent from the symmetric definitions given below. The definitions via universal morphisms are often useful for establishing that a given functor is left or right adjoint, because they are minimalistic in their requirements. They are also intuitively meaningful in that finding a universal morphism is like solving an optimization problem.

Definition via counit-unit adjunction

A **counit-unit adjunction** between two categories C and D consists of two functors $F : C \leftarrow D$ and $G : C \to D$ and two natural transformations

$$\varepsilon : FG \to 1_{\mathcal{C}}$$
$$\eta : 1_{\mathcal{D}} \to GF$$

respectively called the **counit** and the **unit** of the adjunction (terminology from universal algebra*), such that the compositions

$$F \xrightarrow{F\eta} FGF \xrightarrow{\varepsilon F} F$$
$$G \xrightarrow{\eta G} GFG \xrightarrow{G\varepsilon} G$$

are the identity transformations 1_F and 1_G on F and G respectively.

In this situation we say that ***F* is left adjoint to *G*** and ***G* is right adjoint to *F***, and may indicate this relationship by writing $(\varepsilon, \eta) : F \dashv G$, or simply $F \dashv G$.

In equation form, the above conditions on (ε,η) are the **counit-unit equations**

$$1_F = \varepsilon F \circ F\eta$$
$$1_G = G\varepsilon \circ \eta G$$

which mean that for each X in C and each Y in D,

$$1_{FY} = \varepsilon_{FY} \circ F(\eta_Y)$$
$$1_{GX} = G(\varepsilon_X) \circ \eta_{GX}.$$

These equations are useful in reducing proofs about adjoint functors to algebraic manipulations. They are sometimes called the *zig-zag equations* because of the appearance of the corresponding string diagrams. A way to remember them is to first write down the nonsensical equation $1 = \varepsilon \circ \eta$ and then fill in either F or G in one of the two simple ways which make the compositions defined.

Note: The use of the prefix "co" in counit here is not consistent with the terminology of limits and colimits, because a colimit satisfies an *initial* property whereas the counit morphisms will satisfy *terminal* properties, and dually. The term *unit* here is borrowed from the theory of monads where it looks like the insertion of the identity 1 into a monoid.

Definition via hom-set adjunction

A **hom-set adjunction** between two categories C and D consists of two functors $F : C \leftarrow D$ and $G : C \rightarrow D$ and a natural isomorphism

$$\Phi : \hom_C(F-,-) \to \hom_D(-,G-).$$

This specifies a family of bijections

$$\Phi_{Y,X} : \hom_C(FY,X) \to \hom_D(Y,GX).$$

for all objects X in C and Y in D.

In this situation we say that ***F* is left adjoint to *G*** and ***G* is right adjoint to *F***, and may indicate this relationship by writing $\Phi : F \dashv G$, or simply $F \dashv G$.

This definition is a logical compromise in that it is somewhat more difficult to satisfy than the universal morphism definitions, and has fewer immediate implications than the counit-unit definition. It is useful because of its obvious symmetry, and as a stepping-stone between the other definitions.

In order to interpret Φ as a *natural isomorphism*, one must recognize $\hom_C(F-,-)$ and $\hom_D(-,G-)$ as functors. In fact, they are both bifunctors from $D^{\mathrm{op}} \times C$ to **Set** (the category of sets). For details, see the article on hom functors. Explicitly, the naturality of Φ means that for all morphisms $f : X \to X'$ in C and all morphisms $g : Y' \to Y$ in D the following diagram commutes:

$$\begin{tikzcd}
\mathrm{Hom}_C(FY,X) \arrow[r, equal, "\Phi_{Y,X}"] \arrow[d, "\mathrm{Hom}(Fg,f)"'] & \mathrm{Hom}_D(Y,GX) \arrow[d, "\mathrm{Hom}(g,Gf)"] \\
\mathrm{Hom}_C(FY',X') \arrow[r, equal, "\Phi_{Y',X'}"'] & \mathrm{Hom}_D(Y',GX')
\end{tikzcd}$$

The vertical arrows in this diagram are those induced by composition with f and g.

Adjunctions in full

There are hence numerous functors and natural transformations associated with every adjunction, and only a small portion is sufficient to determine the rest.

An *adjunction* between categories C and D consists of

- A functor $F : C \leftarrow D$ called the **left adjoint**
- A functor $G : C \rightarrow D$ called the **right adjoint**
- A natural isomorphism $\Phi : \hom_C(F-,-) \rightarrow \hom_D(-,G-)$
- A natural transformation $\varepsilon : FG \rightarrow 1_C$ called the **counit**
- A natural transformation $\eta : 1_D \rightarrow GF$ called the **unit**

An equivalent formulation, where X denotes any object of C and Y denotes any object of D:

For every C*-morphism* $f : FY \to X$ *there is a unique* D*-morphism* $\Phi_{Y,X}(f) = g : Y \to GX$ *such that the diagrams below commute, and for every* D*-morphism* $g : Y \to GX$ *there is a unique* C*-morphism* $\Phi^{-1}_{X,Y}(g) = f : FY \to X$ *in* C *such that the diagrams below commute:*

$$\begin{tikzcd} F(Y) \arrow[d, "F(g)"'] \arrow[dr, "f"] & \\ FG(X) \arrow[r, "\varepsilon_X"'] & X \end{tikzcd} \qquad \begin{tikzcd} Y \arrow[r, "\eta_Y"] \arrow[dr, "g"'] & GF(Y) \arrow[d, "G(f)"] \\ & G(X) \end{tikzcd}$$

From this assertion, one can recover that:

- The transformations ε, η, and Φ are related by the equations

$$\begin{aligned} f = \Phi^{-1}_{Y,X}(g) = \varepsilon_X \circ F(g) &\quad \in \hom_C(F(Y), X) \\ g = \Phi_{Y,X}(f) = G(f) \circ \eta_Y &\quad \in \hom_D(Y, G(X)) \\ \Phi^{-1}_{GX,X}(1_{GX}) = \varepsilon_X &\quad \in \hom_C(FG(X), X) \\ \Phi_{Y,FY}(1_{FY}) = \eta_Y &\quad \in \hom_D(Y, GF(Y)) \end{aligned}$$

- The transformations ε, η satisfy the counit-unit equations

$$1_F = \varepsilon F \circ F\eta$$
$$1_G = G\varepsilon \circ \eta G$$

- Each pair (GX, ε_X) is a terminal morphism from F to X in C
- Each pair (FY, η_Y) is an initial morphism from Y to G in D

In particular, the equations above allow one to define Φ, ε, and η in terms of any one of the three. However, the adjoint functors F and G alone are in general not sufficient to determine the adjunction. We will demonstrate the equivalence of these situations below.

Universal morphisms induce hom-set adjunction

Given a right adjoint functor $G : C \to D$ in the sense of initial morphisms, we can construct a functor $F : C \leftarrow D$ and hom-set adjunction

$$\Phi : \hom_C(F-,-) \to \hom_D(-,G-)$$

in the following steps:

- For each Y in D, choose an initial morphism (X_Y, η_Y) from Y to G, so we have $\eta_Y \to G(X_Y)$.
- Initiality of these morphisms allows us to construct a unique functor $F : C \leftarrow D$ such that $FY = X_Y$ and $\eta : 1_D \to GF$ is a natural transformation.
- For every morphism $g : Y \to GX$, initiality of (FY, η_Y) means we can let $\Psi_{Y,X}(g)$ be the unique morphism $f : FY \to X$ such that $G(f) \circ \eta_Y = g$.
- The map $\Psi_{Y,X} : \hom_C(FY, X) \leftarrow \hom_D(Y, GX)$ is injective by uniqueness and surjective because we can solve its defining equation for f. It is natural in X because η is natural, and natural in Y because G is a functor. Hence letting $\Phi_{Y,X} = \Psi_{Y,X}^{-1} : \hom_C(FY, X) \to \hom_D(Y, GX)$ gives a hom-set adjunction as required.

A similar argument allows one to construct a hom-set adjunction from the terminal morphisms to a left adjoint functor. (The construction that starts with a right adjoint is slightly more common, since the right adjoint in many adjoint pairs is a trivially defined inclusion or forgetful functor.)

Counit-unit adjunction induces hom-set adjunction

Given functors $F : C \leftarrow D$, $G : C \to D$, and a counit-unit adjunction $(\varepsilon, \eta) : F \dashv G$, we can construct a hom-set adjunction

$$\Phi : \hom_D(F-,-) \to \hom_C(-,G-)$$

in the following steps:

- For each $f : FY \to X$ and each $g : Y \to GX$, define

  $$\Phi_{Y,X}(f) = G(f) \circ \eta_Y$$
  $$\Psi_{Y,X}(g) = \varepsilon_X \circ F(g)$$

  The transformations Φ and Ψ are natural because η and ε are natural.

- Using, in order, that F is a functor, that ε is natural, and the counit-unit equation $1_{FY} = \varepsilon_{FY} \circ F(\eta_Y)$, we obtain

  $$\begin{aligned}\Psi\Phi f &= \varepsilon_X \circ FG(f) \circ F(\eta_Y) \\ &= f \circ \varepsilon_{FY} \circ F(\eta_Y) \\ &= f \circ 1_{FY} = f\end{aligned}$$

  hence $\Psi\Phi$ is the identity transformation.

- Dually, using that G is a functor, that η is natural, and the counit-unit equation $1_{GX} = G(\varepsilon_X) \circ \eta_{GX}$, we obtain

  $$\begin{aligned}\Phi\Psi g &= G(\varepsilon_X) \circ GF(g) \circ \eta_Y \\ &= G(\varepsilon_X) \circ \eta_{GX} \circ g \\ &= 1_{GX} \circ g = g\end{aligned}$$

  hence $\Phi\Psi$ is the identity transformation, so Φ is a natural isomorphism with inverse $\Phi^{-1} = \Psi$.

Hom-set adjunction induces all of the above

Given functors $F : C \leftarrow D$, $G : C \rightarrow D$, and a hom-set adjunction $\Phi : \mathrm{hom}_C(F-,-) \rightarrow \mathrm{hom}_D(-,G-)$, we can construct a counit-unit adjunction

$(\varepsilon, \eta) : F \dashv G$,

which defines families of initial and terminal morphisms, in the following steps:

- Let $\varepsilon_X = \Phi^{-1}_{GX,X}(1_{GX}) \in \mathrm{hom}_C(FGX, X)$ for each *X* in *C*, where $1_{GX} \in \mathrm{hom}_D(GX, GX)$ is the identity morphism.
- Let $\eta_Y = \Phi_{Y,FY}(1_{FY}) \in \mathrm{hom}_D(Y, GFY)$ for each *Y* in *D*, where $1_{FY} \in \mathrm{hom}_C(FY, FY)$ is the identity morphism.
- The bijectivity and naturality of Φ imply that each (GX, ε_X) is a terminal morphism from *X* to *F* in *C*, and each (FY, η_Y) is an initial morphism from *Y* to *G* in *D*.
- The naturality of Φ implies the naturality of ε and η, and the two formulas

  $$\Phi_{Y,X}(f) = G(f) \circ \eta_Y$$
  $$\Phi^{-1}_{Y,X}(g) = \varepsilon_X \circ F(g)$$

  for each *f*: *FY* → *X* and *g*: *Y* → *GX* (which completely determine Φ).
- Substituting *FY* for *X* and $\eta_Y = \Phi_{Y,FY}(1_{FY})$ for *g* in the second formula gives the first counit-unit equation

  $$1_{FY} = \varepsilon_{FY} \circ F(\eta_Y),$$

  and substituting *GX* for *Y* and $\varepsilon_X = \Psi^{-1}_{GX,X}(1_{GX})$ for *f* in the first formula gives the second counit-unit equation

  $$1_{GX} = G(\varepsilon_Y) \circ \eta_{GX}.$$

Historical perspective

Ubiquity of adjoint functors

The idea of an adjoint functor was formulated by Daniel Kan in 1958. Like many of the concepts in category theory, it was suggested by the needs of homological algebra, which was at the time devoted to computations. Those faced with giving tidy, systematic presentations of the subject would have noticed relations such as

hom(*F*(*X*), *Y*) = hom(*X*, *G*(*Y*))

in the category of abelian groups, where *F* was the functor $- \otimes A$ (i.e. take the tensor product with *A*), and *G* was the functor hom(*A*,–). The use of the *equals* sign is an abuse of notation; those two groups are not really identical but there is a way of identifying them that is *natural*. It can be seen to be natural on the basis, firstly, that these are two alternative descriptions of the bilinear mappings from *X* × *A* to *Y*. That is, however, something particular to the case of tensor product. In category theory the 'naturality' of the bijection is subsumed in the concept of a natural isomorphism.

The terminology comes from the Hilbert space idea of adjoint operators *T*, *U* with <*Tx*,*y*> = <*x*,*Uy*>, which is formally similar to the above relation between hom-sets. We say that *F* is *left adjoint* to *G*, and *G* is *right adjoint* to *F*. Note that *G* may have itself a right adjoint that is quite different from *F* (see below for an example). The analogy to adjoint maps of Hilbert spaces can be made precise in certain contexts[1] .

If one starts looking for these adjoint pairs of functors, they turn out to be very common in abstract algebra, and elsewhere as well. The example section below provides evidence of this; furthermore, universal constructions, which may be more familiar to some, give rise to numerous adjoint pairs of functors.

In accordance with the thinking of Saunders Mac Lane, any idea such as adjoint functors that occurs widely enough in mathematics should be studied for its own sake.

Problems formulated with adjoint functors

Mathematicians do not generally need the full adjoint functor concept. Concepts can be judged according to their use in solving problems, as well as for their use in building theories. The tension between these two motivations was especially great during the 1950s when category theory was initially developed. Enter Alexander Grothendieck, who used category theory to take compass bearings in other work — in functional analysis, homological algebra and finally algebraic geometry.

It is probably wrong to say that he promoted the adjoint functor concept in isolation: but recognition of the role of adjunction was inherent in Grothendieck's approach. For example, one of his major achievements was the formulation of Serre duality in relative form — one could say loosely, in a continuous family of algebraic varieties. The entire proof turned on the existence of a right adjoint to a certain functor. This is something undeniably abstract, and non-constructive, but also powerful in its own way.

The case of partial orders

Every partially ordered set can be viewed as a category (with a single morphism between x and y if and only if $x \leq y$). A pair of adjoint functors between two partially ordered sets is called a Galois connection (or, if it is contravariant, an *antitone* Galois connection). See that article for a number of examples: the case of Galois theory of course is a leading one. Any Galois connection gives rise to closure operators and to inverse order-preserving bijections between the corresponding closed elements.

As is the case for Galois groups, the real interest lies often in refining a correspondence to a duality (i.e. *antitone* order isomorphism). A treatment of Galois theory along these lines by Kaplansky was influential in the recognition of the general structure here.

The partial order case collapses the adjunction definitions quite noticeably, but can provide several themes:

- adjunctions may not be dualities or isomorphisms, but are candidates for upgrading to that status
- closure operators may indicate the presence of adjunctions, as corresponding monads (cf. the Kuratowski closure axioms)
- a very general comment of Martin Hyland is that *syntax and semantics* are adjoint: take C to be the set of all logical theories (axiomatizations), and D the power set of the set of all mathematical structures. For a theory T in C, let $F(T)$ be the set of all structures that satisfy the axioms T; for a set of mathematical structures S, let $G(S)$ be the minimal axiomatization of S. We can then say that $F(T)$ is a subset of S if and only if T logically implies $G(S)$: the "semantics functor" F is left adjoint to the "syntax functor" G.
- division is (in general) the attempt to *invert* multiplication, but many examples, such as the introduction of implication in propositional logic, or the ideal quotient for division by ring ideals, can be recognised as the attempt to provide an adjoint.

Together these observations provide explanatory value all over mathematics.

Examples

Free groups (instructive example)

The construction of free groups is an extremely common adjoint construction, and a useful example for making sense of the above details.

Suppose that F : **Grp** ← **Set** is the functor assigning to each set Y the free group generated by the elements of Y, and that G : **Grp** → **Set** is the forgetful functor functor, which assigns to each group X its underlying set. Then F is left adjoint to G:

Terminal morphisms. For each group X, the group FGX is the free group generated freely by GX, the elements of X. Let $\varepsilon_X : FGX \to X$ be the group homomorphism which sends the generators of FGX to the elements of X

they correspond to, which exists by the universal property of free groups. Then each (GX, ε_X) is a terminal morphism from F to X, because any group homomorphism from a free group FZ to X will factor through $\varepsilon_X : FGX \to X$ via a unique set from Z to GX. This means that (F,G) is an adjoint pair.

Initial morphisms. For each set Y, the set GFY is just the underlying set of the free group FY generated by Y. Let $\eta_Y : Y \to GFY$ be the set map given by "inclusion of generators". Then each (FY, η_Y) is an initial morphism from Y to G, because any set map from Y to the underlying set GW of a group will factor through $\eta_Y : Y \to GFY$ via a unique group homomorphism from FY to W. This also means that (F,G) is an adjoint pair.

Hom-set adjunction. Maps from the free group FY to a group X correspond precisely to maps from the set Y to the set GX: each homomorphism from FY to X is fully determined by its action on generators. One can verify directly that this correspondence is a natural transformation, which means it is a hom-set adjunction for the pair (F,G).

Counit-unit adjunction. One can also verify directly that ε and η are natural. Then, a direct verification that they form a counit-unit adjunction $(\varepsilon, \eta) : F \dashv G$ is as follows:

The first counit-unit equation $1_F = \varepsilon F \circ F\eta$ says that for each set Y the composition

$$FY \xrightarrow{F(\eta_Y)} FGFY \xrightarrow{\varepsilon_{FY}} FY$$

should be the identity. The intermediate group $FGFY$ is the free group generated freely by the words of the free group FY. (Think of these words as placed in parentheses to indicate that they are independent generators.) The arrow $F(\eta_Y)$ is the group homomorphism from FY into $FGFY$ sending each generator y of FY to the corresponding word of length one (y) as a generator of $FGFY$. The arrow ε_{FY} is the group homomorphism from $FGFY$ to FY sending each generator to the word of FY it corresponds to (so this map is "dropping parentheses"). The composition of these maps is indeed the identity on FY.

The second counit-unit equation $1_G = G\varepsilon \circ \eta G$ says that for each group X the composition

$$GX \xrightarrow{\eta_{GX}} GFGX \xrightarrow{G(\varepsilon_X)} GX$$

should be the identity. The intermediate set $GFGX$ is just the underlying set of FGX. The arrow η_{GX} is the "inclusion of generators" set map from the set GX to the set $GFGX$. The arrow $G(\varepsilon_X)$ is the set map from $GFGX$ to GX which underlies the group homomorphism sending each generator of FGX to the element of X it corresponds to ("dropping parentheses"). The composition of these maps is indeed the identity on GX.

Free constructions and forgetful functors

Free objects are all examples of a left adjoint to a forgetful functor which assigns to an algebraic object its underlying set. These algebraic free functors have generally the same description of as in the detailed description of the free group situation above.

Diagonal functors and limits

Products, fibred products, equalizers, and kernels are all examples of the categorical notion of a limit. Any limit functor is right adjoint to a corresponding diagonal functor (provided the category has the type of limits in question), and the counit of the adjunction provides the defining maps from the limit object. Below are some specific examples.

- **Products** Let Π : $\mathbf{Grp}^2 \to \mathbf{Grp}$ the functor which assigns to each pair (X_1, X_2) the product group $X_1 \times X_2$, and let Δ : $\mathbf{Grp}^2 \leftarrow \mathbf{Grp}$ be the diagonal functor which assigns to every group X the pair (X, X) in the product category $\mathbf{Grp}^2$. The universal property of the product group shows that Π is right-adjoint to Δ. The counit of this adjunction is the defining pair of projection maps from $X_1 \times X_2$ to X_1 and X_2 which define the limit, and the unit is the *diagonal inclusion* of a group X into $X_1 \times X_2$ (mapping x to (x,x)).

  The cartesian product of sets, the product of rings, the product of topological spaces etc. follow the same pattern; it can also be extended in a straightforward manner to more than just two factors. More generally, any type of limit is right adjoint to a diagonal functor.

- **Kernels.** Consider the category D of homomorphisms of abelian groups. If $f_1 : A_1 \to B_1$ and $f_2 : A_2 \to B_2$ are two objects of D, then a morphism from f_1 to f_2 is a pair (g_A, g_B) of morphisms such that $g_B f_1 = f_2 g_A$. Let $G : D \to \mathbf{Ab}$ be the functor which assigns to each homomorphism its kernel and let $F : \mathbf{D} \leftarrow \mathbf{Ab}$ be the functor which maps the group A to the homomorphism $A \to 0$. Then G is right adjoint to F, which expresses the universal property of kernels. The counit of this adjunction is the defining embedding of a homomorphism's kernel into the homomorphism's domain, and the unit is the morphism identifying a group A with the kernel of the homomorphism $A \to 0$.

  A suitable variation of this example also shows that the kernel functors for vector spaces and for modules are right adjoints. Analogously, one can show that the cokernel functors for abelian groups, vector spaces and modules are left adjoints.

Colimits and diagonal functors

Coproducts, fibred coproducts, coequalizers, and cokernels are all examples of the categorical notion of a colimit. Any colimit functor is left adjoint to a corresponding diagonal functor (provided the category has the type of colimits in question), and the unit of the adjunction provides the defining maps into the colimit object. Below are some specific examples.

- **Coproducts.** If $F : \mathbf{Ab} \leftarrow \mathbf{Ab}^2$ assigns to every pair (X_1, X_2) of abelian groups their direct sum, and if $G : \mathbf{Ab} \to \mathbf{Ab}^2$ is the functor which assigns to every abelian group Y the pair (Y, Y), then F is left adjoint to G, again a consequence of the universal property of direct sums. The unit of this adjoint pair is the defining pair of inclusion maps from X_1 and X_2 into the direct sum, and the counit is the additive map from the direct sum of (X,X) to back to X (sending an element (a,b) of the direct sum to the element $a+b$ of X).

  Analogous examples are given by the direct sum of vector spaces and modules, by the free product of groups and by the disjoint union of sets.

Further examples

In algebra

- **Adjoining an identity to a rng.** This example was discussed in the motivation section above. Given a rng R, a multiplicative identity element can be added by taking Rx$\mathbf{Z}$ and defining a $\mathbf{Z}$-bilinear product with (r,0)(0,1) = (0,1)(r,0) = (r,0), (r,0)(s,0) = (rs,0), (0,1)(0,1) = (0,1). This constructs a left adjoint to the functor taking a ring to the underlying rng.
- **Ring extensions.** Suppose R and S are rings, and $\rho : R \to S$ is a ring homomorphism. Then S can be seen as a (left) R-module, and the tensor product with S yields a functor $F : R\text{-}\mathbf{Mod} \to S\text{-}\mathbf{Mod}$. Then F is left adjoint to the forgetful functor $G : S\text{-}\mathbf{Mod} \to R\text{-}\mathbf{Mod}$.
- **Tensor products.** If R is a ring and M is a right R module, then the tensor product with M yields a functor $F : R\text{-}\mathbf{Mod} \to \mathbf{Ab}$. The functor $G : \mathbf{Ab} \to R\text{-}\mathbf{Mod}$, defined by $G(A) = \hom_{\mathbf{Z}}(M,A)$ for every abelian group A, is a right adjoint to F.
- **From monoids and groups to rings** The integral monoid ring construction gives a functor from monoids to rings. This functor is left adjoint to the functor that associates to a given ring its underlying multiplicative monoid. Similarly, the integral group ring construction yields a functor from groups to rings, left adjoint to the functor that assigns to a given ring its group of units. One can also start with a field K and consider the category of K-algebras instead of the category of rings, to get the monoid and group rings over K.
- **Field of fractions.** Consider the category $\mathbf{Dom}_m$ of integral domains with injective morphisms. The forgetful functor $\mathbf{Field} \to \mathbf{Dom}_m$ from fields has a left adjoint - it assigns to every integral domain its field of fractions.

- **Polynomial rings**. Let $\mathbf{Ring}_*$ be the category of pointed commutative rings with unity (pairs (A,a) where A is a ring, $a \in A$ and morphisms preserve the distinguished elements). The forgetful functor G:$\mathbf{Ring}_* \to \mathbf{Ring}$ has a left adjoint - it assigns to every ring R the pair (R[x],x) where R[x] is the polynomial ring with coefficients from R.
- **Abelianization**. Consider the inclusion functor $G : \mathbf{Ab} \to \mathbf{Grp}$ from the category of abelian groups to category of groups. It has a left adjoint called abelianization which assigns to every group G the quotient group $G^{ab}=G/[G,G]$.
- **The Grothendieck group**. In K-theory, the point of departure is to observe that the category of vector bundles on a topological space has a commutative monoid structure under direct sum. One may make an abelian group out of this monoid, the Grothendieck group, by formally adding an additive inverse for each bundle (or equivalence class). Alternatively one can observe that the functor that for each group takes the underlying monoid (ignoring inverses) has a left adjoint. This is a once-for-all construction, in line with the third section discussion above. That is, one can imitate the construction of negative numbers; but there is the other option of an existence theorem. For the case of finitary algebraic structures, the existence by itself can be referred to universal algebra, or model theory; naturally there is also a proof adapted to category theory, too.
- **Frobenius reciprocity** in the representation theory of groups: see induced representation. This example foreshadowed the general theory by about half a century.

In topology

- **A functor with a left and a right adjoint.** Let G be the functor from topological spaces to sets that associates to every topological space its underlying set (forgetting the topology, that is). G has a left adjoint F, creating the discrete space on a set Y, and a right adjoint H creating the trivial topology on Y.
- **Suspensions and loop spaces** Given topological spaces X and Y, the space $[SX, Y]$ of homotopy classes of maps from the suspension SX of X to Y is naturally isomorphic to the space $[X, \Omega Y]$ of homotopy classes of maps from X to the loop space ΩY of Y. This is an important fact in homotopy theory.
- **Stone-Čech compactification.** Let **KHaus** be the category of compact Hausdorff spaces and $G : \mathbf{KHaus} \to \mathbf{Top}$ be the forgetful functor to the category of topological spaces. Then G has a left adjoint $F : \mathbf{Top} \to \mathbf{KHaus}$, the Stone–Čech compactification. The counit of this adjoint pair yields a continuous map from every topological space X into its Stone-Čech compactification. This map is an embedding (i.e. injective, continuous and open) if and only if X is a Tychonoff space.
- **Direct and inverse images of sheaves** Every continuous map $f : X \to Y$ between topological spaces induces a functor f_* from the category of sheaves (of sets, or abelian groups, or rings...) on X to the corresponding category of sheaves on Y, the *direct image functor*. It also induces a functor f^{-1} from the category of sheaves of abelian groups on Y to the category of sheaves of abelian groups on X, the *inverse image functor*. f^{-1} is left adjoint to f_*. Here a more subtle point is that the left adjoint for coherent sheaves will differ from that for sheaves (of sets).
- **Soberification.** The article on Stone duality describes an adjunction between the category of topological spaces and the category of sober spaces that is known as soberification. Notably, the article also contains a detailed description of another adjunction that prepares the way for the famous duality of sober spaces and spatial locales, exploited in pointless topology.

In category theory

- **A series of adjunctions.** The functor π_0 which assigns to a category its sets of connected components is left-adjoint to the functor D which assigns to a set the discrete category on that set. Moreover, D is left-adjoint to the object functor U which assigns to each category its set of objects, and finally U is left-adjoint to A which assigns to each set the antidiscrete category on that set.
- **Exponential object**. In a cartesian closed category the endofunctor $C \to C$ given by $-\times A$ has a right adjoint $-^A$.

In categorical logic

- **quantification** Any morphism $f : X \to Y$ in a category with pullbacks induces a monotonous map $f^* : \mathrm{Sub}(Y) \to \mathrm{Sub}(X)$ acting by pullbacks (A monotonous map is a functor if we consider the preorders as categories). If this functor has a left/right adjoint, the adjoint is called $\exists_f$ and $\forall_f$, respectively.

  In the category of sets, if we choose subsets as the canonical subobjects, then these functions are given by:

$$\begin{aligned}
(T \subseteq Y) \ &\mapsto\ f^*(T) = f^{-1}[T] \\
(S \subseteq X) \ &\mapsto\ \exists_f X = \{\ y \in Y \ \mid\ \exists x \in f^{-1}[\{y\}], x \in S\ \} = f[S] \\
(S \subseteq X) \ &\mapsto\ \forall_f X = \{\ y \in Y \ \mid\ \forall x \in f^{-1}[\{y\}], x \in S\ \}
\end{aligned}$$

Properties

Uniqueness of adjoints

If the functor $F : C \leftarrow D$ has two right adjoints G and G', then G and G' are naturally isomorphic. The same is true for left adjoints.

Conversely, if F is left adjoint to G, and G is naturally isomorphic to G' then F is also left adjoint to G'. More generally, if $\langle F, G, \varepsilon, \eta\rangle$ is an adjunction (with counit-unit (ε,η)) and

$\sigma : F \to F'$

$\tau : G \to G'$

are natural isomorphisms then $\langle F', G', \varepsilon', \eta'\rangle$ is an adjunction where

$$\begin{aligned}
\eta' &= (\tau * \sigma) \circ \eta \\
\varepsilon' &= \varepsilon \circ (\sigma^{-1} * \tau^{-1}).
\end{aligned}$$

Here $\circ$ denotes vertical composition of natural transformations, and $*$ denotes horizontal composition.

Composition

Adjunctions can be composed in a natural fashion. Specifically, if $\langle F, G, \varepsilon, \eta\rangle$ is an adjunction between C and D and $\langle F', G', \varepsilon', \eta'\rangle$ is an adjunction between D and E then the functor

$$F' \circ F : \mathcal{C} \leftarrow \mathcal{E}$$

is left adjoint to

$$G \circ G' : \mathcal{C} \to \mathcal{E}.$$

More precisely, there is an adjunction between $F'\ F$ and $G\ G'$ with unit and counit given by the compositions:

$$1_{\mathcal{E}} \xrightarrow{\eta} GF \xrightarrow{G\eta' F} GG'F'F$$

$$F'FGG' \xrightarrow{F'\varepsilon G'} F'G' \xrightarrow{\varepsilon'} 1_{\mathcal{C}}.$$

This new adjunction is called the **composition** of the two given adjunctions.

One can then form a category whose objects are all small categories and whose morphisms are adjunctions.

Adjoints preserve limits

The most important property of adjoints is their continuity: every functor that has a left adjoint (and therefore *is* a right adjoint) is *continuous* (i.e. commutes with limits in the category theoretical sense); every functor that has a right adjoint (and therefore *is* a left adjoint) is *cocontinuous* (i.e. commutes with colimits).

Since many common constructions in mathematics are limits or colimits, this provides a wealth of information. For example:

- applying a right adjoint functor to a product of objects yields the product of the images;
- applying a left adjoint functor to a coproduct of objects yields the coproduct of the images;
- every right adjoint functor is left exact;
- every left adjoint functor is right exact.

Additivity

If C and D are preadditive categories and $F : C \leftarrow D$ is an additive functor with a right adjoint $G : C \rightarrow D$, then G is also an additive functor and the hom-set bijections

$$\Phi_{Y,X} : \hom_{\mathcal{C}}(FY, X) \cong \hom_{\mathcal{D}}(Y, GX)$$

are, in fact, isomorphisms of abelian groups. Dually, if G is additive with a left adjoint F, then F is also additive.

Moreover, if both C and D are additive categories (i.e. preadditive categories with all finite biproducts), then any pair of adjoint functors between them are automatically additive.

General existence theorem

Not every functor $G : C \rightarrow D$ admits a left adjoint. If C is a complete category, then the functors with left adjoints can be characterized by the **adjoint functor theorem** of Peter J. Freyd: G has a left adjoint if and only if it is continuous and a certain smallness condition is satisfied: for every object Y of D there exists a family of morphisms

$$f_i : Y \rightarrow G(X_i)$$

where the indices i come from a *set* I, not a *proper class*, such that every morphism

$$h : Y \rightarrow G(X)$$

can be written as

$$h = G(t) \circ f_i$$

for some i in I and some morphism

$$t : X_i \rightarrow X \text{ in } C.$$

An analogous statement characterizes those functors with a right adjoint.

Relationship to other categorical concepts

Universal constructions

As stated earlier, an adjunction between categories C and D gives rise to a family of universal morphisms, one for each object in C and one for each object in D. Conversely, if there exists a universal morphism to a functor $G : C \rightarrow D$ from every object of D, then G has a left adjoint.

However, universal constructions are more general than adjoint functors: a universal construction is like an optimization problem; it gives rise to an adjoint pair if and only if this problem has a solution for every object of D (equivalently, every object of C).

Equivalences of categories

If a functor F: $C \rightarrow D$ is one half of an equivalence of categories then it is the left adjoint in an adjoint equivalence of categories, i.e. an adjunction whose unit and counit are isomorphisms.

Every adjunction $\langle F, G, \varepsilon, \eta\rangle$ extends an equivalence of certain subcategories. Define C_1 as the full subcategory of C consisting of those objects X of C for which ε_X is an isomorphism, and define D_1 as the full subcategory of D consisting of those objects Y of D for which η_Y is an isomorphism. Then F and G can be restricted to D_1 and C_1 and yield inverse equivalences of these subcategories.

In a sense, then, adjoints are "generalized" inverses. Note however that a right inverse of F (i.e. a functor G such that FG is naturally isomorphic to 1_D) need not be a right (or left) adjoint of F. Adjoints generalize *two-sided* inverses.

Monads

Every adjunction $\langle F, G, \varepsilon, \eta\rangle$ gives rise to an associated monad $\langle T, \eta, \mu\rangle$ in the category D. The functor

$$T : \mathcal{D} \to \mathcal{D}$$

is given by $T = GF$. The unit of the monad

$$\eta : 1_{\mathcal{D}} \to T$$

is just the unit η of the adjunction and the multiplication transformation

$$\mu : T^2 \to T$$

is given by $\mu = G\varepsilon F$. Dually, the triple $\langle FG, \varepsilon, F\eta G\rangle$ defines a comonad in D.

Every monad arises from some adjunction—in fact, typically from many adjunctions—in the above fashion. Two constructions, called the category of Eilenberg–Moore algebras and the Kleisli category are two extremal solutions to the problem of constructing an adjunction that gives rise to a given monad.

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- Adámek, Jiří; Horst Herrlich, and George E. Strecker (1990). *Abstract and Concrete Categories* (http://katmat.math.uni-bremen.de/acc/acc.pdf). John Wiley & Sons. ISBN 0-471-60922-6.
- Mac Lane, Saunders (1998). *Categories for the Working Mathematician*. Graduate Texts in Mathematics **5** ((2nd ed.) ed.). Springer-Verlag. ISBN 0-387-98403-8.

External links

- Adjunctions (http://www.youtube.com/view_play_list?p=54B49729E5102248) Seven short lectures on adjunctions.

Natural transformations

In category theory, a branch of mathematics, a **natural transformation** provides a way of transforming one functor into another while respecting the internal structure (i.e. the composition of morphisms) of the categories involved. Hence, a natural transformation can be considered to be a "morphism of functors". Indeed this intuition can be formalized to define so-called functor categories. Natural transformations are, after categories and functors, one of the most basic notions of category theory and consequently appear in the majority of its applications.

Definition

If F and G are functors between the categories C and D, then a **natural transformation** η from F to G associates to every object X in C a morphism $\eta_X : F(X) \to G(X)$ in D called the **component** of η at X, such that for every morphism $f : X \to Y$ in C we have:

$$\eta_Y \circ F(f) = G(f) \circ \eta_X$$

This equation can conveniently be expressed by the commutative diagram

$$\begin{tikzcd} F(X) \arrow[r, "F(f)"] \arrow[d, "\eta_X"'] & F(Y) \arrow[d, "\eta_Y"] \\ G(X) \arrow[r, "G(f)"'] & G(Y) \end{tikzcd}$$

If both F and G are contravariant, the horizontal arrows in this diagram are reversed. If η is a natural transformation from F to G, we also write $\eta : F \to G$ or $\eta : F \Rightarrow G$. This is also expressed by saying the family of morphisms $\eta_X : F(X) \to G(X)$ is **natural** in X.

If, for every object X in C, the morphism η_X is an isomorphism in D, then η is said to be a **natural isomorphism** (or sometimes **natural equivalence** or **isomorphism of functors**). Two functors F and G are called *naturally isomorphic* or simply *isomorphic* if there exists a natural isomorphism from F to G.

An **infranatural transformation** η from F to G is simply a family of morphisms η_X: $F(X) \to G(X)$. Thus a natural transformation is an infranatural transformation for which $\eta_Y \circ F(f) = G(f) \circ \eta_X$ for every morphism $f : X \to Y$. The **naturalizer** of η, nat(η), is the largest subcategory of C containing all the objects of C on which η restricts to a natural transformation.

Examples

A worked example

Statements such as

> "Every group is naturally isomorphic to its opposite group"

abound in modern mathematics. We will now give the precise meaning of this statement as well as its proof. Consider the category **Grp** of all groups with group homomorphisms as morphisms. If $(G,*)$ is a group, we define its opposite group $(G^{op},*^{op})$ as follows: G^{op} is the same set as G, and the operation $*^{op}$ is defined by $a*^{op}b = b*a$. All multiplications in G^{op} are thus "turned around". Forming the opposite group becomes a (covariant!) functor from **Grp** to **Grp** if we define $f^{op} = f$ for any group homomorphism f: $G \to H$. Note that f^{op} is indeed a group homomorphism from G^{op} to H^{op}:

$$f^{op}(a*^{op}b) = f(b*a) = f(b)*f(a) = f^{op}(a)*^{op}f^{op}(b).$$

The content of the above statement is:

"The identity functor $\mathrm{Id}_{\mathbf{Grp}}$: **Grp** → **Grp** is naturally isomorphic to the opposite functor $^{\mathrm{op}}$: **Grp** → **Grp**."

To prove this, we need to provide isomorphisms $\eta_G : G \to G^{\mathrm{op}}$ for every group G, such that the above diagram commutes. Set $\eta_G(a) = a^{-1}$. The formulas $(ab)^{-1} = b^{-1}\,a^{-1}$ and $(a^{-1})^{-1} = a$ show that η_G is a group homomorphism which is its own inverse. To prove the naturality, we start with a group homomorphism $f : G \to H$ and show $\eta_H \circ f = f^{\mathrm{op}} \circ \eta_G$, i.e. $(f(a))^{-1} = f^{\mathrm{op}}(a^{-1})$ for all a in G. This is true since $f^{\mathrm{op}} = f$ and every group homomorphism has the property $(f(a))^{-1} = f(a^{-1})$.

Further examples

If K is a field, then for every vector space V over K we have a "natural" injective linear map $V \to V^{**}$ from the vector space into its double dual. These maps are "natural" in the following sense: the double dual operation is a functor, and the maps are the components of a natural transformation from the identity functor to the double dual functor.

Every finite dimensional vector space is also isomorphic to its dual space. But this isomorphism relies on an arbitrary choice of basis, and is not natural, though there is an infranatural transformation. More generally, any vector spaces with the same dimensionality are isomorphic, but not naturally so. (Note however that if the space has a nondegenerate bilinear form, then there *is* a natural isomorphism between the space and its dual. Here the space is viewed as an object in the category of vector spaces and transposes of maps.)

Consider the category **Ab** of abelian groups and group homomorphisms. For all abelian groups X, Y and Z we have a group isomorphism

$$\mathrm{Hom}(X \otimes Y, Z) \to \mathrm{Hom}(X, \mathrm{Hom}(Y, Z)).$$

These isomorphisms are "natural" in the sense that they define a natural transformation between the two involved functors $\mathbf{Ab} \times \mathbf{Ab}^{\mathrm{op}} \times \mathbf{Ab}^{\mathrm{op}} \to \mathbf{Ab}$.

Natural transformations arise frequently in conjunction with adjoint functors. Indeed, adjoint functors are defined by a certain natural isomorphism. Additionally, every pair of adjoint functors comes equipped with two natural transformations (generally not isomorphisms) called the *unit* and *counit*.

Operations with natural transformations

If $\eta : F \to G$ and $\varepsilon : G \to H$ are natural transformations between functors $F,G,H : C \to D$, then we can compose them to get a natural transformation $\varepsilon\eta : F \to H$. This is done componentwise: $(\varepsilon\eta)_X = \varepsilon_X \eta_X$. This "vertical composition" of natural transformation is associative and has an identity, and allows one to consider the collection of all functors $C \to D$ itself as a category (see below under Functor categories).

Natural transformations also have a "horizontal composition". If $\eta : F \to G$ is a natural transformation between functors $F,G : C \to D$ and $\varepsilon : J \to K$ is a natural transformation between functors $J, K : D \to E$, then the composition of functors allows a composition of natural transformations $\eta\varepsilon : JF \to KG$. This operation is also associative with identity, and the identity coincides with that for vertical composition. The two operations are related by an identity which exchanges vertical composition with horizontal composition.

If $\eta : F \to G$ is a natural transformation between functors $F, G : C \to D$, and $H : D \to E$ is another functor, then we can form the natural transformation $H\eta : HF \to HG$ by defining

$$(H\eta)_X = H\eta_X.$$

If on the other hand $K : B \to C$ is a functor, the natural transformation $\eta K : FK \to GK$ is defined by

$$(\eta K)_X = \eta_{K(X)}.$$

Functor categories

If C is any category and I is a small category, we can form the functor category C^I having as objects all functors from I to C and as morphisms the natural transformations between those functors. This forms a category since for any functor F there is an identity natural transformation $1_F : F \to F$ (which assigns to every object X the identity morphism on $F(X)$) and the composition of two natural transformations (the "vertical composition" above) is again a natural transformation.

The isomorphisms in C^I are precisely the natural isomorphisms. That is, a natural transformation $\eta : F \to G$ is a natural isomorphism if and only if there exists a natural transformation $\varepsilon : G \to F$ such that $\eta\varepsilon = 1_G$ and $\varepsilon\eta = 1_F$.

The functor category C^I is especially useful if I arises from a directed graph. For instance, if I is the category of the directed graph • → •, then C^I has as objects the morphisms of C, and a morphism between $\varphi : U \to V$ and $\psi : X \to Y$ in C^I is a pair of morphisms $f : U \to X$ and $g : V \to Y$ in C such that the "square commutes", i.e. $\psi f = g \varphi$.

More generally, one can build the 2-category **Cat** whose

- 0-cells (objects) are the small categories,
- 1-cells (arrows) between two objects C and D are the functors from C to D ,
- 2-cells between two 1-cells (functors) $F : C \to D$ and $G : C \to D$ are the natural transformations from F to G .

The horizontal and vertical compositions are the compositions between natural transformations described previously. A functor category C^I is then simply a hom-category in this category (smallness issues aside).

Yoneda lemma

If X is an object of a locally small category C, then the assignment $Y \mapsto \mathrm{Hom}_C(X, Y)$ defines a covariant functor $F_X : C \to$ **Set**. This functor is called *representable* (more generally, a representable functor is any functor naturally isomorphic to this functor for an appropriate choice of X). The natural transformations from a representable functor to an arbitrary functor $F : C \to$ **Set** are completely known and easy to describe; this is the content of the Yoneda lemma.

Historical notes

Saunders Mac Lane, one of the founders of category theory, is said to have remarked, "I didn't invent categories to study functors; I invented them to study natural transformations." Just as the study of groups is not complete without a study of homomorphisms, so the study of categories is not complete without the study of functors. The reason for Mac Lane's comment is that the study of functors is itself not complete without the study of natural transformations.

The context of Mac Lane's remark was the axiomatic theory of homology. Different ways of constructing homology could be shown to coincide: for example in the case of a simplicial complex the groups defined directly, and those of the singular theory, would be isomorphic. What cannot easily be expressed without the language of natural transformations is how homology groups are compatible with morphisms between objects, and how two equivalent homology theories not only have the same homology groups, but also the same morphisms between those groups.

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Algebraic category

In mathematics, specifically universal algebra, a **variety of algebras** is the class of all algebraic structures of a given signature satisfying a given set of identities. Equivalently, a variety is a class of algebraic structures of the same signature which is closed under the taking of homomorphic images, subalgebras and (direct) products. In the context of category theory, a variety of algebras is usually called a **finitary algebraic category**.

A **covariety** is the class of all coalgebraic structures of a given signature.

A variety of algebras should not be confused with an algebraic variety. Intuitively, a variety of algebras is an equationally defined **collection of algebras**, while an algebraic variety is an equationally defined **collection of elements from a single algebra**. The two are named alike by analogy, but they are formally quite distinct and their theories have little in common.

Birkhoff's theorem

Garrett Birkhoff proved equivalent the two definitions of variety given above, a result of fundamental importance to universal algebra and known as **Birkhoff's theorem** or as the **HSP theorem**. **H**, **S**, and **P** stand, respectively, for the closure operations of homomorphism, subalgebra, and product.

An equational class for some signature Σ is the collection of all models, in the sense of model theory, that satisfy some set *E* of *equations*, asserting equality between terms. A model *satisfies* these equations if they are true in the model for any valuation of the variables. The equations in *E* are then said to be identities of the model. Examples of such identities are the commutative law, characterizing commutative algebras, and the absorption law, characterizing lattices.

It is simple to see that the class of algebras satisfying some set of equations will be closed under the HSP operations. Proving the converse —classes of algebras closed under the HSP operations must be equational— is much harder.

Examples

The class of all semigroups forms a variety of algebras of signature (2). A sufficient defining equation is the associative law:

$$x(yz) = (xy)z.$$

It satisfies the HSP closure requirement, since any homomorphic image, any subset closed under multiplication and any direct product of semigroups is also a semigroup.

The class of groups forms a class of algebras of signature (2,1,0), the three operations being respectively *multiplication*, *inversion* and *identity*. Any subset of a group closed under multiplication, under inversion and under identity (i.e. containing the identity) forms a subgroup. Likewise, the collection of groups is closed under homomorphic image and under direct product. Applying Birkhoff's theorem, this is sufficient to tell us that the groups form a variety, and so it should be defined by a collection of identities. In fact, the familiar axioms of associativity, inverse and identity form one suitable set of identities:

$$x(yz) = (xy)z$$
$$1x = x1 = x$$
$$xx^{-1} = x^{-1}x = 1.$$

A **subvariety** is a subclass of a variety, closed under the operations H, S, P. Notice that although every group is a semigroup, the class of groups does **not** form a subvariety of the variety of semigroups. This is because not every sub**semi**group of a group is a group.

The class of abelian groups, considered again with signature (2,1,0), also has the HSP closure properties. It forms a subvariety of the variety of groups, and can be defined equationally by the three group axioms above together with the commutativity law:

$$xy = yx.$$

Variety of finite algebras

Since varieties are closed under arbitrary cartesian products, all non-trivial varieties contain infinite algebras. It follows that the theory of varieties is of limited use in the study of finite algebras, where one must often apply techniques particular to the finite case. With this in mind, attempts have been made to develop a finitary analogue of the theory of varieties.

A **variety of finite algebras**, sometimes called a **pseudovariety**, is usually defined to be a class of finite algebras of a given signature, closed under the taking of homomorphic images, subalgebras and finitary direct products. There is no general finitary counterpart to Birkhoff's theorem, but in many cases the introduction of a more complex notion of equations allows similar results to be derived.

Pseudovarieties are of particular importance in the study of finite semigroups and hence in formal language theory. Eilenberg's theorem, often referred to as the *variety theorem* describes a natural correspondence between varieties of regular languages and pseudovarieties of finite semigroups.

Category theory

If *A* is a finitary algebraic category, then the forgetful functor

$$U : A \to \mathbf{Set}$$

is monadic. Even more, it is *strictly monadic*, in that the comparison functor

$$K : A \to \mathbf{Set}^{\mathbb{T}}$$

is an isomorphism (and not just an equivalence).[1] Here, $\mathbf{Set}^{\mathbb{T}}$ is the Eilenberg–Moore category on $\mathbf{Set}$. In general, one says a category is an **algebraic category** if it is monadic over $\mathbf{Set}$. This is a more general notion than "finitary algebraic category" (the notion of "variety" used in universal algebra) because it admits such categories as **CABA** (complete atomic Boolean algebras) and **CSLat** (complete semilattices) whose signatures include infinitary operations. In those two cases the signature is large, meaning that it forms not a set but a proper class, because its operations are of unbounded arity. The algebraic category of sigma algebras also has infinitary operations, but their arity is countable whence its signature is small (forms a set).

See also

- Quasivariety

Notes

[1] Saunders Mac Lane, *Categories for the Working Mathematician*, Springer. *(See p. 152)*

References

Two monographs available free online:

- Burris, Stanley N., and H.P. Sankappanavar, H. P., 1981. *A Course in Universal Algebra.* (http://www.thoralf.uwaterloo.ca/htdocs/ualg.html) Springer-Verlag. ISBN 3-540-90578-2.
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Domain theory

Domain theory is a branch of mathematics that studies special kinds of partially ordered sets (posets) commonly called **domains**. Consequently, domain theory can be considered as a branch of order theory. The field has major applications in computer science, where it is used to specify denotational semantics, especially for functional programming languages. Domain theory formalizes the intuitive ideas of approximation and convergence in a very general way and has close relations to topology. An alternative important approach to denotational semantics in computer science is that of metric spaces.

Motivation and intuition

The primary motivation for the study of domains, which was initiated by Dana Scott in the late 1960s, was the search for a denotational semantics of the lambda calculus. In this formalism, one considers "functions" specified by certain terms in the language. In a purely syntactic way, one can go from simple functions to functions that take other functions as their input arguments. Using again just the syntactic transformations available in this formalism, one can obtain so called fixed point combinators (the best-known of which is the Y combinator); these, by definition, have the property that $f(\mathbf{Y}(f)) = \mathbf{Y}(f)$ for all functions f.

To formulate such a denotational semantics, one might first try to construct a *model* for the lambda calculus, in which a genuine (total) function is associated with each lambda term. Such a model would formalize a link between the lambda calculus as a purely syntactic system and the lambda calculus as a notational system for manipulating concrete mathematical functions. Unfortunately, such a model cannot exist, for if it did, it would have to contain a genuine, total function that corresponds to the combinator **Y**, that is, a function that computes a fixed point of an *arbitrary* input function f. There can be no such function for **Y**, because some functions (for example, the *successor function*) do not have a fixed point. At best, the genuine function corresponding to **Y** would have to be a partial function, necessarily undefined at some inputs.

Scott got around this difficulty by formalizing a notion of "partial" or "incomplete" information to represent computations that have not yet returned a result. This was modeled by considering, for each domain of computation (e.g. the natural numbers), an additional element that represents an *undefined* output, i.e. the "result" of a computation that never ends. In addition, the domain of computation is equipped with an *ordering relation*, in which the "undefined result" is the least element.

The important step to find a model for the lambda calculus is to consider only those functions (on such a partially ordered set) which are guaranteed to have least fixed points. The set of these functions, together with an appropriate ordering, is again a "domain" in the sense of the theory. But the restriction to a subset of all available functions has another great benefit: it is possible to obtain domains that contain their own function spaces, i.e. one gets functions that can be applied to themselves.

Beside these desirable properties, domain theory also allows for an appealing intuitive interpretation. As mentioned above, the domains of computation are always partially ordered. This ordering represents a hierarchy of information or knowledge. The higher an element is within the order, the more specific it is and the more information it contains. Lower elements represent incomplete knowledge or intermediate results.

Computation then is modeled by applying monotone functions repeatedly on elements of the domain in order to refine a result. Reaching a fixed point is equivalent to finishing a calculation. Domains provide a superior setting for these ideas since fixed points of monotone functions can be guaranteed to exist and, under additional restrictions, can be approximated from below.

A guide to the formal definitions

In this section, the central concepts and definitions of domain theory will be introduced. The above intuition of domains being *information orderings* will be emphasized to motivate the mathematical formalization of the theory. The precise formal definitions are to be found in the dedicated articles for each concept. A list of general order-theoretic definitions which include domain theoretic notions as well can be found in the order theory glossary. The most important concepts of domain theory will nonetheless be introduced below.

Directed sets as converging specifications

As mentioned before, domain theory deals with partially ordered sets to model a domain of computation. The goal is to interpret the elements of such an order as *pieces of information* or *(partial) results of a computation*, where elements that are higher in the order extend the information of the elements below them in a consistent way. From this simple intuition it is already clear that domains often do not have a greatest element, since this would mean that there is an element that contains the information of *all* other elements - a rather uninteresting situation.

A concept that plays an important role in the theory is the one of a **directed subset** of a domain, i.e. of a non-empty subset of the order in which each two elements have some upper bound that is an element of this subset. In view of our intuition about domains, this means that every two pieces of information within the directed subset are *consistently* extended by some other element in the subset. Hence we can view directed sets as *consistent specifications*, i.e. as sets of partial results in which no two elements are contradictory. This interpretation can be compared with the notion of a convergent sequence in analysis, where each element is more specific than the preceding one. Indeed, in the theory of metric spaces, sequences play a role that is in many aspects analogous to the role of directed sets in domain theory.

Now, as in the case of sequences, we are interested in the *limit* of a directed set. According to what was said above, this would be an element that is the most general piece of information which extends the information of all elements of the directed set, i.e. the unique element that contains *exactly* the information that was present in the directed set - and nothing more. In the formalization of order theory, this is just the **least upper bound** of the directed set. As in the case of limits of sequences, least upper bounds of directed sets do not always exist.

Naturally, one has a special interest in those domains of computations in which all consistent specifications *converge*, i.e. in orders in which all directed sets have a least upper bound. This property defines the class of **directed complete partial orders**, or **dcpo** for short. Indeed, most considerations of domain theory do only consider orders that are at least directed complete.

From the underlying idea of partially specified results as representing incomplete knowledge, one derives another desirable property: the existence of a **least element**. Such an element models that state of no information - the place where most computations start. It also can be regarded as the output of a computation that does not return any result at all.

Computations and domains

Now that we have some basic formal descriptions of what a domain of computation should be, we can turn to the computations themselves. Clearly, these have to be functions, taking inputs from some computational domain and returning outputs in some (possibly different) domain. However, one would also expect that the output of a function will contain more information when the information content of the input is increased. Formally, this means that we want a function to be **monotonic**.

When dealing with **dcpos**, one might also want computations to be compatible with the formation of limits of a directed set. Formally, this means that, for some function *f*, the image *f*(*D*) of a directed set *D* (i.e. the set of the images of each element of *D*) is again directed and has as a least upper bound the image of the least upper bound of *D*. One could also say that *f preserves directed suprema*. Also note that, by considering directed sets of two elements, such a function also has to be monotonic. These properties give rise to the notion of a **Scott-continuous** function. Since this often is not ambiguous one also may speak of *continuous functions*.

Approximation and finiteness

Domain theory is a purely *qualitative* approach to modeling the structure of information states. One can say that something contains more information, but the amount of additional information is not specified. Yet, there are some situations in which one wants to speak about elements that are in a sense much simpler (or much more incomplete) than a given state of information. For example, in the natural subset-inclusion ordering on some powerset, any infinite element (i.e. set) is much more "informative" than any of its *finite* subsets.

If one wants to model such a relationship, one may first want to consider the induced strict order $<$ of a domain with order $\leq$. However, while this is a useful notion in the case of total orders, it does not tell us much in the case of partially ordered sets. Considering again inclusion-orders of sets, a set is already strictly smaller than another, possibly infinite, set if it contains just one less element. One would, however, hardly agree that this captures the notion of being "much simpler".

Way-below relation

A more elaborate approach leads to the definition of the so-called **order of approximation**, which is more suggestively also called the **way-below relation**. An element x is *way below* an element y, if, for every directed set D with supremum such that

$$y \leq \sup D,$$

there is some element d in D such that

$$x \leq d.$$

Then one also says that x *approximates* y and writes

$$x \ll y.$$

This does imply that

$$x \leq y,$$

since the singleton set $\{y\}$ is directed. For an example, in an ordering of sets, an infinite set is way above any of its finite subsets. On the other hand, consider the directed set (in fact: the chain) of finite sets

$$\{0\}, \{0, 1\}, \{0, 1, 2\}, \ldots$$

Since the supremum of this chain is the set of all natural numbers **N**, this shows that no infinite set is way below **N**.

However, being way below some element is a *relative* notion and does not reveal much about an element alone. For example, one would like to characterize finite sets in an order-theoretic way, but even infinite sets can be way below some other set. The special property of these **finite** elements x is that they are way below themselves, i.e.

$$x \ll x.$$

An element with this property is also called **compact**. Yet, such elements do not have to be "finite" nor "compact" in any other mathematical usage of the terms. The notation is nonetheless motivated by certain parallels to the respective notions in set theory and topology. The compact elements of a domain have the important special property that they cannot be obtained as a limit of a directed set in which they did not already occur.

Many other important results about the way-below relation support the claim that this definition is appropriate to capture many important aspects of a domain.

Bases of domains

The previous thoughts raise another question: is it possible to guarantee that all elements of a domain can be obtained as a limit of much simpler elements? This is quite relevant in practice, since we cannot compute infinite objects but we may still hope to approximate them arbitrarily closely.

More generally, we would like to restrict to a certain subset of elements as being sufficient for getting all other elements as least upper bounds. Hence, one defines a **base** of a poset *P* as being a subset *B* of *P*, such that, for each *x* in *P*, the set of elements in *B* that are way below *x* contains a directed set with supremum *x*. The poset *P* is a *continuous poset* if it has some base. Especially, *P* itself is a base in this situation. In many applications, one restricts to continuous (d)cpos as a main object of study.

Finally, an even stronger restriction on a partially ordered set is given by requiring the existence of a base of *compact* elements. Such a poset is called **algebraic**. From the viewpoint of denotational semantics, algebraic posets are particularly well-behaved, since they allow for the approximation of all elements even when restricting to finite ones. As remarked before, not every finite element is "finite" in a classical sense and it may well be that the finite elements constitute an uncountable set.

In some cases, however, the base for a poset is countable. In this case, one speaks of an **ω-continuous** poset. Accordingly, if the countable base consists entirely of finite elements, we obtain an order that is **ω-algebraic**.

Special types of domains

A simple special case of a domain is known as an **elementary** or **flat domain**. This consists of a set of incomparable elements, such as the integers, along with a single "bottom" element considered smaller than all other elements.

One can obtain a number of other interesting special classes of ordered structures that could be suitable as "domains". We already mentioned continuous posets and algebraic posets. More special versions of both are continuous and algebraic cpos. Adding even further completeness properties one obtains continuous lattices and algebraic lattices, which are just complete lattices with the respective properties. For the algebraic case, one finds broader classes of posets which are still worth studying: historically, the Scott domains were the first structures to be studied in domain theory. Still wider classes of domains are constituted by SFP-domains, L-domains, and bifinite domains.

All of these classes of orders can be cast into various categories of dcpos, using functions which are monotone, Scott-continuous, or even more specialized as morphisms. Finally, note that the term *domain* itself is not exact and thus is only used as an abbreviation when a formal definition has been given before or when the details are irrelevant.

Important results

A poset *D* is a dcpo if and only if each chain in *D* has a supremum. However, directed sets are strictly more powerful than chains.

If *f* is a continuous function on a poset *D* then it has a least fixed point, given as the least upper bound of all finite iterations of *f* on the least element *0*: $\bigvee_{n \text{ in } \mathbf{N}} f^n(0)$.

Generalizations

- Synthetic domain theory [1]
- Topological domain theory [2]
- A continuity space is a generalization of metric spaces and posets, that can be used to unify the notions of metric spaces and domains.

Further reading

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Enriched category theory

In category theory and its applications to mathematics, an **enriched category** is a category whose hom-sets are replaced by objects from some other category, in a well-behaved manner.

Definition

We define here what it means for **C** to be an **enriched category** over a monoidal category $(\mathbf{M}, \otimes, I)$.

The following structures are required:

- Let Ob(**C**) be a set (or proper class). An element of Ob(**C**) is called an *object* of **C**.
- For each pair (A,B) of objects of **C**, let Hom(A,B) be an object of **M**, called the *hom-object* of A and B.
- For each object A of **C**, let id_A be a morphism in **M** from I to Hom(A,A), called the *identity morphism* of A.
- For each triple (A,B,C) of objects of **C**, let

$$\circ : \mathrm{Hom}(B,C) \otimes \mathrm{Hom}(A,B) \to \mathrm{Hom}(A,C)$$

be a morphism in **M** called the *composition* morphism of A, B, and C.

The following axioms are required:

- Associativity: Given objects A, B, C, and D of **C**, we can go from Hom(C,D) ⊗ Hom(B,C) ⊗ Hom(A,B) to Hom(A,D) in two ways, depending on which composition we do first. These must give the same result.

$$\begin{tikzcd}
(\mathrm{Hom}(C,D) \otimes \mathrm{Hom}(B,C)) \otimes \mathrm{Hom}(A,B) \arrow[r, "\alpha"] \arrow[d, "\circ\otimes 1"'] & \mathrm{Hom}(C,D) \otimes (\mathrm{Hom}(B,C) \otimes \mathrm{Hom}(A,B)) \arrow[r, "1\otimes\circ"] & \mathrm{Hom}(C,D) \otimes \mathrm{Hom}(A,C) \arrow[d, "\circ"] \\
\mathrm{Hom}(B,D) \otimes \mathrm{Hom}(A,B) \arrow[rr, "\circ"'] & & \mathrm{Hom}(A,D)
\end{tikzcd}$$

- Left identity: Given objects A and B of **C**, we can go from $I \otimes \mathrm{Hom}(A,B)$ to just $\mathrm{Hom}(A,B)$ in two ways, either by using id_A on I and then using composition, or by simply using the fact that I is an identity for $\otimes$ in **M**. These must give the same result.
- Right identity: Given objects A and B of **C**, we can go from $\mathrm{Hom}(A,B) \otimes I$ to just $\mathrm{Hom}(A,B)$ in two ways, either by using id_B on I and then using composition, or by simply using the fact that I is an identity for $\otimes$ in **M**. These must give the same result.

Given the above, **C** (consisting of all the structures listed above) is a category enriched over **M**.

Examples

The most straightforward example is to take **M** to be a category of sets, with the Cartesian product for the monoidal operation. Then **C** is nothing but an ordinary category. If **M** is the category of small sets, then **C** is a locally small category, because the hom-sets will all be small. Similarly, if **M** is the category of finite sets, then **C** is a locally finite category.

If **M** is the category **2** with Ob(**2**) = {0,1}, a single nonidentity morphism (from 0 to 1), and ordinary multiplication of numbers as the monoidal operation, then **C** can be interpreted as a preordered set. Specifically, $A \le B$ iff $\mathrm{Hom}(A,B) = 1$.

If **M** is a category of pointed sets with smash product for the monoidal operation, then **C** is a category with zero morphisms. Specifically, the zero morphism from A to B is the special point in the pointed set $\mathrm{Hom}(A,B)$.

If **M** is a category of abelian groups with tensor product as the monoidal operation, then **C** is a preadditive category.

Relationship with monoidal functors

If there is a monoidal functor from a monoidal category **M** to a monoidal category **N**, then any category enriched over **M** can be reinterpreted as a category enriched over **N**. Every monoidal category **M** has a monoidal functor **M**(I, –) to the category of sets, so any enriched category has an underlying ordinary category. In many examples (such as those above) this functor is faithful, so a category enriched over **M** can be described as an ordinary category with certain additional structure or properties.

Enriched functors

An **enriched functor** is the appropriate generalization of the notion of a functor to enriched categories. Enriched functors are then maps between enriched categories which respect the enriched structure.

If C and D are **M**-categories (that is, categories enriched over monoidal category **M**), an **M**-enriched functor $T\colon C \to D$ is a map which assigns to each object of C an object of D and for each pair of objects a and b in C provides a morphism in **M** $T_{ab}\colon C(a,b) \to D(T(a),T(b))$ between the hom-objects of C and D (which are objects in **M**), satisfying enriched versions of the axioms of a functor, viz preservation of identity and composition.

Because the hom-objects need not be sets in an enriched category, one cannot speak of a particular morphism. There is no longer any notion of an identity morphism, nor of a particular composition of two morphisms. Instead, morphisms from the unit to a hom-object should be thought of as selecting an identity and morphisms from the monoidal product should be thought of as composition. The usual functorial axioms are replaced with corresponding commutative diagrams involving these morphisms.

In detail, one has that the diagram

$$\begin{tikzcd} & I \arrow[dl, "\mathrm{id}_a"'] \arrow[dr, "\mathrm{id}_b"] & \\ C(a,a) \arrow[rr, "T_{aa}"] & & D(T(a),T(a)) \end{tikzcd}$$

commutes, which amounts to the equation

$$T_{aa} \circ \mathrm{id}_a = \mathrm{id}_{T(a)},$$

where I is the unit object of **M**. This is analogous to the rule $F(\mathrm{id}_a) = \mathrm{id}_{F(a)}$ for ordinary functors. Additionally, one demands that the diagram

$$\begin{tikzcd} C(b,c) \otimes C(a,b) \arrow[r, "\circ"] \arrow[d, "T_{bc} \otimes T_{ab}"'] & C(a,c) \arrow[d, "T_{ac}"] \\ D(T(b),T(c)) \otimes D(T(a),T(b)) \arrow[r, "\circ"'] & D(T(a),T(c)) \end{tikzcd}$$

commute, which is analogous to the rule $F(fg)=F(f)F(g)$ for ordinary functors.

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Topos

In mathematics, a **topos** (plural "topoi" or "toposes") is a type of category that behaves like the category of sheaves of sets on a topological space. For a discussion of the history of topos theory, see the article Background and genesis of topos theory.

Grothendieck topoi (topoi in geometry)

Since the introduction of sheaves into mathematics in the 1940s a major theme has been to study a space by studying sheaves on a space. This idea was expounded by Alexander Grothendieck by introducing the notion of a **topos**. The main utility of this notion is in the abundance of situations in mathematics where topological intuition is very effective but an honest topological space is lacking; it is sometimes possible to find a topos formalizing the intuition. The greatest single success of this programmatic idea to date has been the introduction of the étale topos of a scheme.

Equivalent formulations

Let C be a category. A theorem of Giraud states that the following are equivalent:

- There is a small category D and an inclusion $C \hookrightarrow \mathrm{Presh}(D)$ that admits a finite-limit-preserving left adjoint.
- C is the category of sheaves on a Grothendieck site.
- C satisfies Giraud's axioms, below.

A category with these properties is called a "(Grothendieck) topos". Here Presh(D) denotes the category of contravariant functors from D to the category of sets; such a contravariant functor is frequently called a presheaf.

Giraud's axioms

Giraud's axioms for a category C are:

- C has a small set of generators, and admits all small colimits. Furthermore, colimits commute with fiber products.
- Sums in C are disjoint. In other words, the fiber product of X and Y over their sum is the initial object in C.
- All equivalence relations in C are effective.

The last axiom needs the most explanation. If X is an object of C, an equivalence relation R on X is a map $R \to X \times X$ in C such that all the maps $\mathrm{Hom}(Y,R) \to \mathrm{Hom}(Y,X) \times \mathrm{Hom}(Y,X)$ are equivalence relations of sets. Since C has colimits we may form the coequalizer of the two maps $R \to X$; call this X/R. The equivalence relation is effective if the canonical map

$$R \to X \times_{X/R} X$$

is an isomorphism.

Examples

Giraud's theorem already gives "sheaves on sites" as a complete list of examples. Note, however, that nonequivalent sites often give rise to equivalent topoi. As indicated in the introduction, sheaves on ordinary topological spaces motivate many of the basic definitions and results of topos theory.

The category of sets is an important special case: it plays the role of a point in topos theory. Indeed, a set may be thought of as a sheaf on a point.

More exotic examples, and the *raison d'être* of topos theory, come from algebraic geometry. To a scheme and even a stack one may associate an étale topos, an fppf topos, a Nisnevich topos...

Counterexamples

Topos theory is, in some sense, a generalization of classical point-set topology. One should therefore expect to see old and new instances of pathological behavior. For instance, there is an example due to Pierre Deligne of a nontrivial topos that has no points (see below).

Geometric morphisms

If X and Y are topoi, a *geometric morphism* u: $X \to Y$ is a pair of adjoint functors (u^*,u_*) such that u^* preserves finite limits. Note that u^* automatically preserves colimits by virtue of having a right adjoint.

By Freyd's adjoint functor theorem, to give a geometric morphism $X \to Y$ is to give a functor u^*: $Y \to X$ that preserves finite limits and all small colimits. Thus geometric morphisms between topoi may be seen as analogues of maps of locales.

If X and Y are topological spaces and u is a continuous map between them, then the pullback and pushforward operations on sheaves yield a geometric morphism between the associated topoi.

Points of topoi

A point of a topos X is a geometric morphism from the topos of sets to X.

If X is an ordinary space and x is a point of X, then the functor that takes a sheaf F to its stalk F_x has a right adjoint (the "skyscraper sheaf" functor), so an ordinary point of X also determines a topos-theoretic point. These may be constructed as the pullback-pushforward along the continuous map x: $1 \to X$.

Essential geometric morphisms

A geometric morphism (u^*,u_*) is *essential* if u^* has a further left adjoint $u_!$, or equivalently (by the adjoint functor theorem) if u^* preserves not only finite but all small limits.

Ringed topoi

A **ringed topos** is a pair (X,R), where X is a topos and R is a commutative ring object in X. Most of the constructions of ringed spaces go through for ringed topoi. The category of R-module objects in X is an abelian category with enough injectives. A more useful abelian category is the subcategory of quasi-coherent R-modules: these are R-modules that admit a presentation.

Another important class of ringed topoi, besides ringed spaces, are the etale topoi of Deligne-Mumford stacks.

Homotopy theory of topoi

Michael Artin and Barry Mazur associated to any topos a pro-simplicial set. Using this inverse system of simplicial sets one may *sometimes* associate to a homotopy invariant in classical topology an inverse system of invariants in topos theory.

The pro-simplicial set associated to the etale topos of a scheme is a pro-finite simplicial set. Its study is called étale homotopy theory.

Elementary toposes (toposes in logic)

Introduction

A traditional axiomatic foundation of mathematics is set theory, in which all mathematical objects are ultimately represented by sets (even functions which map between sets). More recent work in category theory allows this foundation to be generalized using toposes; each topos completely defines its own mathematical framework. The category of sets forms a familiar topos, and working within this topos is equivalent to using traditional set theoretic mathematics. But one could instead choose to work with many alternative toposes. A standard formulation of the axiom of choice makes sense in any topos, and there are toposes in which it is invalid. Constructivists will be interested to work in a topos without the law of excluded middle. If symmetry under a particular group G is of importance, one can use the topos consisting of all G-sets.

It is also possible to encode an algebraic theory, such as the theory of groups, as a topos. The individual models of the theory, i.e. the groups in our example, then correspond to functors from the encoding topos to the category of sets that respect the topos structure.

Formal definition

When used for foundational work a topos will be defined axiomatically; set theory is then treated as a special case of topos theory. Building from category theory, there are multiple equivalent definitions of a topos. The following has the virtue of being concise, if not illuminating:

A topos is a category which has the following two properties:

- All limits taken over finite index categories exist.
- Every object has a power object.

From this one can derive that

- All colimits taken over finite index categories exist.
- The category has a subobject classifier.
- Any two objects have an exponential object.
- The category is cartesian closed.

In many applications, the role of the subobject classifier is pivotal, whereas power objects are not. Thus some definitions reverse the roles of what is defined and what is derived.

Explanation

A topos as defined above can be understood as a cartesian closed category for which the notion of subobject of an object has an elementary or first-order definition. This notion, as a natural categorical abstraction of the notions of subset of a set, subgroup of a group, and more generally subalgebra of any algebraic structure, predates the notion of topos. It is definable in any category, not just toposes, in second-order language, i.e. in terms of classes of morphisms instead of individual morphisms, as follows. Given two monics m, n from respectively Y and Z to X, we say that $m \leq n$ when there exists a morphism $p: Y \to Z$ for which $np = m$, inducing a preorder on monics to X. When $m \leq n$ and $n \leq m$ we say that m and n are equivalent. The subobjects of X are the resulting equivalence classes of the monics to it.

In a topos "subobject" becomes, at least implicitly, a first-order notion, as follows.

As noted above, a topos is a category C having all finite limits and hence in particular the empty limit or final object 1. It is then natural to treat morphisms of the form $x: 1 \to X$ as *elements* $x \in X$. Morphisms $f: X \to Y$ thus correspond to functions mapping each element $x \in X$ to the element $fx \in Y$, with application realized by composition.

One might then think to define a subobject of X as an equivalence class of monics $m: X' \to X$ having the same image or range $\{mx | x \in X'\}$. The catch is that two or more morphisms may correspond to the same function, that is, we cannot assume that C is concrete in the sense that the functor $C(1,-): C \to$ **Set** is faithful. For example the category **Grph** of graphs and their associated homomorphisms is a topos whose final object 1 is the graph with one vertex and one edge (a self-loop), but is not concrete because the elements $1 \to G$ of a graph G correspond only to the self-loops and not the other edges, nor the vertices without self-loops. Whereas the second-order definition makes G and its set of self-loops (with their vertices) distinct subobjects of G (unless every edge is, and every vertex has, a self-loop), this image-based one does not. This can be addressed for the graph example and related examples via the Yoneda Lemma as described in the Examples section below, but this then ceases to be first-order. Toposes provide a more abstract, general, and first-order solution.

As noted above a topos C has a subobject classifier Ω, namely an object of C with an element $t \in \Omega$, the *generic subobject* of C, having the property that every monic m: $X' \to X$ arises as a pullback of the generic subobject along a unique morphism f: $X \to \Omega$, as per Figure 1. Now the pullback of a monic is a monic, and all elements including t are monics since there is only one morphism to 1 from any given object, whence the pullback of t along f: $X \to \Omega$ is a monic. The monics to X are therefore in bijection with the pullbacks of t along morphisms from X to Ω. The latter morphisms partition the monics into equivalence classes each determined by a morphism f: $X \to \Omega$, the characteristic morphism of that class, which we take to be the subobject of X characterized or named by f.

$$\begin{tikzcd} X' \arrow[r, "m"] \arrow[d, "!"'] & X \arrow[d, "f"] \\ 1 \arrow[r, "t"] & \Omega \end{tikzcd}$$

Figure 1. *m* as a pullback of the generic subobject *t* along *f*.

All this applies to any topos, whether or not concrete. In the concrete case, namely $C(1,-)$ faithful, for example the category of sets, the situation reduces to the familiar behavior of functions. Here the monics m: $X' \to X$ are exactly the injections (one-one functions) from X' to X, and those with a given image $\{mx | x \in X'\}$ constitute the subobject of X corresponding to the morphism f: $X \to \Omega$ for which $f^{-1}(t)$ is that image. The monics of a subobject will in general have many domains, all of which however will be in bijection with each other.

To summarize, this first-order notion of subobject classifier implicitly defines for a topos the same equivalence relation on monics to X as had previously been defined explicitly by the second-order notion of subobject for any category. The notion of equivalence relation on a class of morphisms is itself intrinsically second-order, which the definition of topos neatly sidesteps by explicitly defining only the notion of subobject *classifier* Ω, leaving the notion of subobject of X as an implicit consequence characterized (and hence namable) by its associated morphism f: $X \to \Omega$.

Further examples

If C is a small category, then the functor category $\mathbf{Set}^C$ (consisting of all covariant functors from C to sets, with natural transformations as morphisms) is a topos. For instance, the category **Grph** of graphs of the kind permitting multiple directed edges between two vertices is a topos. A graph consists of two sets, an edge set and a vertex set, and two functions s,t between those sets, assigning to every edge e its source $s(e)$ and target $t(e)$. **Grph** is thus equivalent to the functor category $\mathbf{Set}^C$, where C is the category with two objects E and V and two morphisms s,t: $E \to V$ giving respectively the source and target of each edge.

The categories of finite sets, of finite G-sets (actions of a group G on a finite set), and of finite graphs are also toposes.

The Yoneda Lemma asserts that C^{op} embeds in $\mathbf{Set}^C$ as a full subcategory. In the graph example the embedding represents C^{op} as the subcategory of $\mathbf{Set}^C$ whose two objects are V' as the one-vertex no-edge graph and E' as the two-vertex one-edge graph (both as functors), and whose two nonidentity morphisms are the two graph homomorphisms from V' to E' (both as natural transformations). The natural transformations from V' to an arbitrary graph (functor) G constitute the vertices of G while those from E' to G constitute its edges. Although $\mathbf{Set}^C$, which we can identify with **Grph**, is not made concrete by either V' or E' alone, the functor U: $\mathbf{Grph} \to \mathbf{Set}^2$ sending object G to the pair of sets ($\mathbf{Grph}(V',G)$, $\mathbf{Grph}(E',G)$) and morphism h: $G \to H$ to the pair of functions ($\mathbf{Grph}(V',h)$, $\mathbf{Grph}(E',h)$) is faithful. That is, a morphism of graphs can be understood as a *pair* of functions, one mapping the vertices and the other the edges, with application still realized as composition but now with multiple sorts of *generalized* elements. This shows that the traditional concept of a concrete category as one whose objects have an underlying set can be generalized to cater for a wider range of toposes by allowing an object to have multiple underlying sets, that is, to be multisorted.

References

Some gentle papers

- John Baez: "Topos theory in a nutshell. [1]" A gentle introduction.
- Steven Vickers: "Toposes pour les nuls [2]" and "Toposes pour les vraiment nuls. [3]" Elementary and even more elementary introductions to toposes as generalized spaces.
- Illusie, Luc, "What is a ... topos?" [4], *Notices of the AMS*

The following texts are easy-paced introductions to toposes and the basics of category theory. They should be suitable for those knowing little mathematical logic and set theory, even non-mathematicians.

- F. William Lawvere and Stephen H. Schanuel (1997) *Conceptual Mathematics: A First Introduction to Categories*. Cambridge University Press. An "introduction to categories for computer scientists, logicians, physicists, linguists, etc." (cited from cover text).
- F. William Lawvere and Robert Rosebrugh (2003) *Sets for Mathematics*. Cambridge University Press. Introduces the foundations of mathematics from a categorical perspective.

Grothendieck foundational work on toposes:

- Grothendieck and Verdier: *Théorie des topos et cohomologie étale des schémas* (known as SGA4)". New York/Berlin: Springer, ??. (Lecture notes in mathematics, 269–270)

The following monographs include an introduction to some or all of topos theory, but do not cater primarily to beginning students. Listed in (perceived) order of increasing difficulty.

- Colin McLarty (1992) *Elementary Categories, Elementary Toposes*. Oxford Univ. Press. A nice introduction to the basics of category theory, topos theory, and topos logic. Assumes very few prerequisites.
- Robert Goldblatt (1984) *Topoi, the Categorial Analysis of Logic* (Studies in logic and the foundations of mathematics, 98). North-Holland. A good start. Reprinted 2006 by Dover Publications, and available online [5] at Robert Goldblatt's homepage. [6]
- John Lane Bell (2005) *The Development of Categorical Logic*. Handbook of Philosophical Logic, Volume 12. Springer. Version available online [7] at John Bell's homepage. [8]
- Saunders Mac Lane and Ieke Moerdijk (1992) *Sheaves in Geometry and Logic: a First Introduction to Topos Theory*. Springer Verlag. More complete, and more difficult to read.
- Michael Barr and Charles Wells (1985) *Toposes, Triples and Theories*. Springer Verlag. Corrected online version at http://www.cwru.edu/artsci/math/wells/pub/ttt.html [9]. More concise than *Sheaves in Geometry and Logic*, but hard on beginners.

Reference works for experts, less suitable for first introduction

- Francis Borceux (1994) *Handbook of Categorical Algebra 3: Categories of Sheaves*, Volume 52 of the *Encyclopedia of Mathematics and its Applications*. Cambridge University Press. The third part of "Borceux' remarkable magnum opus", as Johnstone has labelled it. Still suitable as an introduction, though beginners may find it hard to recognize the most relevant results among the huge amount of material given.
- Peter T. Johnstone (1977) *Topos Theory*, L. M. S. Monographs no. 10. Academic Press. ISBN 0123878500. For a long time the standard compendium on topos theory. However, even Johnstone describes this work as "far too hard to read, and not for the faint-hearted."
- Peter T. Johnstone (2002) *Sketches of an Elephant: A Topos Theory Compendium*. Oxford Science Publications. As of early 2010, two of the scheduled three volumes of this overwhelming compendium were available.

Books that target special applications of topos theory

- Maria Cristina Pedicchio and Walter Tholen, eds. (2004) *Categorical Foundations: Special Topics in Order, Topology, Algebra, and Sheaf Theory*. Volume 97 of the *Encyclopedia of Mathematics and its Applications*.

Cambridge University Press. Includes many interesting special applications.

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[2] http://www.cs.bham.ac.uk/~sjv/papers.php
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Descent (category theory)

In mathematics, the idea of **descent** has come to stand for a very general idea, extending the intuitive idea of 'gluing' in topology. Since the topologists' glue is actually the use of equivalence relations on topological spaces, the theory starts with some ideas on identification.

A sophisticated theory resulted. It was a tribute to the efforts to use category theory to get around the alleged 'brutality' of imposing equivalence relations within geometric categories. One outcome was the eventual definition adopted in topos theory of geometric morphism, to get the correct notion of surjectivity.

Descent of vector bundles

The case of the construction of vector bundles from data on a disjoint union of topological spaces is a straightforward place to start.

Suppose X is a topological space covered by open sets X_i. Let Y be the disjoint union of the X_i, so that there is a natural mapping

$$p : Y \to X.$$

We think of Y as 'above' X, with the X_i projection 'down' onto X. With this language, *descent* implies a vector bundle on Y (so, a bundle given on each X_i), and our concern is to 'glue' those bundles V_i, to make a single bundle V on X. What we mean is that V should, when restricted to X_i, give back V_i, up to a bundle isomorphism.

The data needed is then this: on each overlap

$$X_{ij},$$

intersection of X_i and X_j, we'll require mappings

$$f_{ij}$$

to use to identify V_i and V_j there, fiber by fiber. Further the f_{ij} must satisfy conditions based on the reflexive, symmetric and transitive properties of an equivalence relation (gluing conditions). For example the composition

$$f_{ij} o f_{jk} = f_{ik}$$

for transitivity (and choosing apt notation). The f_{ii} should be identity maps and hence the symmetry becomes invertibility of f_{ij} (so that it is fiberwise an isomorphism).

These are indeed standard conditions in fiber bundle theory (see transition function). One important application to note is *change of fiber*: if the f_{ij} are all you need to make a bundle, then there are many ways to make an associated bundle. That is, we can take essentially same f_{ij}, acting on various different fibers.

Another major point is the relation with the chain rule: the discussion of the way there of constructing tensor fields can be summed up as 'once you learn to descend the tangent bundle, for which transitivity is the Jacobian chain rule, the rest is just 'naturality of tensor constructions'.

To move closer towards the abstract theory we need to interpret the disjoint union of the

$$X_{ij}$$

now as

$$Y \times_X Y,$$

the fiber product (here an equalizer) of two copies of the projection p. The bundles on the X_{ij} that we must control are actually V' and V'', the pullbacks to the fiber of V via the two different projection maps to X.

Therefore by going to a more abstract level one can eliminate the combinatorial side (that is, leave out the indices) and get something that makes sense for p not of the special form of covering with which we began. This then allows a category theory approach: what remains to do is to re-express the gluing conditions.

History

The ideas were developed in the period 1955-1965 (which was roughly the time at which the requirements of algebraic topology were met but those of algebraic geometry were not). From the point of view of abstract category theory the work of comonads of Beck was a summation of those ideas; see Beck's monadicity theorem.

The difficulties of algebraic geometry with passage to the quotient are acute. The urgency (to put it that way) of the problem for the geometers accounts for the title of the 1959 Grothendieck seminar *TDTE* on *theorems of descent and techniques of existence* (see FGA) connecting the descent question with the representable functor question in algebraic geometry in general, and the moduli problem in particular.

Further reading

Angelo Vistoli, http://arxiv.org/abs/math.AG/0412512

Stack (descent theory)

In mathematics a **stack** is a concept used to formalise some of the main constructions of descent theory.

Descent theory is concerned with generalisations of situations where geometrical objects (such as vector bundles on topological spaces) can be "glued together" when they are isomorphic (in a compatible way) when restricted to intersections of the sets in an open covering of a space. In more general set-up the restrictions are replaced with general pull-backs, and fibred categories form the right framework to discuss the possibility of such "glueing". The intuitive meaning of a stack is that it is a fibred category such that "all possible glueings work". The specification of glueings requires a definition of coverings with regard to which the glueings can be considered. It turns out that the general language for describing these coverings is that of a Grothendieck topology- Thus a stack is formally given as a fibred category over another *base* category, where the base has a Grothendieck topology and where the fibred category satisfies a few axioms that ensure existence and uniqueness of certain glueings with respect to the Grothendieck topology.

Archetypical examples include the stack of vector bundles on topological spaces, the stack of quasi-coherent sheaves on schemes (with respect to the fpqc-topology and weaker topologies) and the stack of affine schemes on a base scheme (again with respect to the fpqc topology or a weaker one).

Stacks are the underlying structure of algebraic stacks, which are a way to generalise schemes and algebraic spaces and which are particularly useful in studying moduli spaces. The concept of stacks has its origin in the definition of effective descent data in Grothendieck (1959). The theory was further developed by Grothendieck and Giraud (1964) and Giraud (1971); the name stack (*champ* in the original French) together with the eventual definition appears to have been introduced in the latter work.

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External links

- Stacks [1] and descent [2] on nLab.

References

[1] http://www.ncatlab.org/nlab/show/stack

[2] http://www.ncatlab.org/nlab/show/descent

Categorical logic

Categorical logic is a branch of category theory within mathematics, adjacent to mathematical logic but more notable for its connections to theoretical computer science. In broad terms, categorical logic represents both syntax and semantics by a category, and an interpretation by a functor. The categorical framework provides a rich conceptual background for logical and type-theoretic constructions. The subject has been recognisable in these terms since around 1970.

Overview

There are three important themes in the categorical approach to logic:

- Categorical semantics. Categorical logic introduces the notion of *structure valued in a category* C with the classical model theoretic notion of a structure appearing in the particular case where C is the category of sets and functions. This notion has proven useful when the set-theoretic notion of a model lacks generality and/or is inconvenient. R.A.G. Seely's modeling of various impredicative theories, such as system F is an example of the usefulness of categorical semantics.
- Internal languages. This can be seen as a formalization and generalization of proof by diagram chasing. One defines a suitable internal language naming relevant constituents of a category, and then applies categorical semantics to turn assertions in a logic over the internal language into corresponding categorical statements. This has been most successful in the theory of toposes, where the internal language of a topos together with the semantics of intuitionistic higher-order logic in a topos enables one to reason about the objects and morphisms of a topos "as if they were sets and functions". This has been successful in dealing with toposes that have "sets" with properties incompatible with classical logic. A prime example is Dana Scott's model of untyped lambda calculus in terms of objects that retract onto their own function space. Another is the Moggi-Hyland model of system F by an internal full subcategory of the effective topos of Martin Hyland.
- Term-model constructions. In many cases, the categorical semantics of a logic provide a basis for establishing a correspondence between theories in the logic and instances of an appropriate kind of category. A classic example is the correspondence between theories of βη-equational logic over simply typed lambda calculus and cartesian closed categories. Categories arising from theories via term-model constructions can usually be characterized up to equivalence by a suitable universal property. This has enabled proofs of meta-theoretical properties of some logics by means of an appropriate categorical algebra. For instance, Freyd gave a proof of the existence and disjunction properties of intuitionistic logic this way.

Historical perspective

Categorical logic originated with Bill Lawvere's *Functorial Semantics of Algebraic Theories* (1963), and *Elementary Theory of the Category of Sets* (1964). Lawvere recognised the Grothendieck topos, introduced in algebraic topology as a generalised space, as a generalisation of the category of sets (*Quantifiers and Sheaves* (1970)). With Myles Tierney, Lawvere then developed the notion of elementary topos, thus establishing the fruitful field of topos theory, which provides a unified categorical treatment of the syntax and semantics of higher-order predicate logic. The resulting logic is formally intuitionistic. Andre Joyal is credited, in the term Kripke–Joyal semantics, with the observation that the sheaf models for predicate logic, provided by topos theory, generalise Kripke semantics. Joyal and others applied these models to study higher-order concepts such as the real numbers in the intuitionistic setting.

An analogous development was the link between the simply typed lambda calculus and cartesian-closed categories (Lawvere, Lambek, Scott), which provided a setting for the development of domain theory. Less expressive theories, from the mathematical logic viewpoint, have their own category theory counterparts. For example the concept of an algebraic theory leads to Gabriel–Ulmer duality. The view of categories as a generalisation unifying syntax and semantics has been productive in the study of logics and type theories for applications in computer science.

The founders of elementary topos theory were Lawvere and Tierney. Lawvere's writings, sometimes couched in a philosophical jargon, isolated some of the basic concepts as adjoint functors (which he explained as 'objective' in a Hegelian sense, not without some justification). A subobject classifier is a strong property to ask of a category, since with cartesian closure and finite limits it gives a topos (axiom bashing shows how strong the assumption is). Lawvere's further work in the 1960s gave a theory of attributes, which in a sense is a subobject theory more in sympathy with type theory. Major influences subsequently have been Martin-Löf type theory from the direction of logic, type polymorphism and the calculus of constructions from functional programming, linear logic from proof theory, game semantics and the projected synthetic domain theory. The abstract categorical idea of fibration has been much applied.

To go back historically, the major irony here is that in large-scale terms, intuitionistic logic had reappeared in mathematics, in a central place in the Bourbaki–Grothendieck program, a generation after the messy Hilbert–Brouwer controversy had ended, with Hilbert the apparent winner. Bourbaki, or more accurately Jean Dieudonné, having laid claim to the legacy of Hilbert and the Göttingen school including Emmy Noether, had revived intuitionistic logic's credibility (although Dieudonné himself found Intuitionistic Logic ludicrous), as the logic of an arbitrary topos, where classical logic was that of 'the' topos of sets. This was one consequence, certainly unanticipated, of Grothendieck's relative point of view; and not lost on Pierre Cartier, one of the broadest of the core group of French mathematicians around Bourbaki and IHES. Cartier was to give a Séminaire Bourbaki exposition of intuitionistic logic.

In an even broader perspective, one might take category theory to be to the mathematics of the second half of the twentieth century, what measure theory was to the first half. It was Kolmogorov who applied measure theory to probability theory, the first convincing (if not the only) axiomatic approach. Kolmogorov was also a pioneer writer in the early 1920s on the formulation of intuitionistic logic, in a style entirely supported by the later categorical logic approach (again, one of the formulations, not the only one; the realizability concept of Stephen Kleene is also a serious contender here). Another route to categorical logic would therefore have been through Kolmogorov, and this is one way to explain the protean Curry–Howard isomorphism.

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Books and handbook chapters

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Further reading

- Michael Makkai and Gonzalo E. Reyes, 1977, *First order categorical logic*
- Lambek, J. and Scott, P. J., 1986. *Introduction to Higher Order Categorical Logic*. Fairly accessible introduction, but somewhat dated. The categorical approach to higher-order logics over polymorphic and dependent types was developed largely after this book was published.
- Jacobs, Bart (1999). *Categorical Logic and Type Theory* [2]. Studies in Logic and the Foundations of Mathematics 141. North Holland, Elsevier. ISBN 0-444-50170-3. A comprehensive monograph written by a computer scientist; it covers both first-order and higher-order logics, and also polymorphic and dependent types. The focus is on fibred category as universal tool in categorical logic, which is necessary in dealing with polymorphic and dependent types. According to P.T. Johnstone, this approach is unwieldy for simple types.
- P.T. Johnstone, 2002, *Sketches of an Elephant*, part D (vol 2) covers both first-order and higher-order logics, but not dependent or polymorphic types, considered by the author of interest mainly to computer science. Because it avoids polymorphic and dependent types, the categorical approach is easier to present in therms of a syntactic category just as in Lambek's book, but *Sketches* includes more recent developments, like .
- John Lane Bell (2005) *The Development of Categorical Logic*. Handbook of Philosophical Logic, Volume 12. Springer. Version available online [7] at John Bell's homepage. [8]
- Jean-Pierre Marquis and Gonzalo E. Reyes (2009). *The History of Categorical Logic 1963–1977*.[3]

External links

- Categorical Logic [4] lecture notes by Steve Awodey [5]

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[1] http://citeseerx.ist.psu.edu/viewdoc/summary?doi=10.1.1.42.9970
[2] http://www.cs.ru.nl/B.Jacobs/CLT/bookinfo.html
[3] http://www.webdepot.umontreal.ca/Usagers/marquisj/MonDepotPublic/HistofCatLog.pdf
[4] http://www.andrew.cmu.edu/user/awodey/catlog/
[5] http://www.andrew.cmu.edu/user/awodey/

Timeline of category theory and related mathematics

This is a **timeline of category theory and related mathematics**. Its scope ('related mathematics') is taken as:

- Categories of abstract algebraic structures including representation theory and universal algebra;
- Homological algebra;
- Homotopical algebra;
- Topology using categories, including algebraic topology, categorical topology, quantum topology, low dimensional topology;
- Categorical logic and set theory in the categorical context such as algebraic set theory;
- Foundations of mathematics building on categories, for instance topos theory;
- Abstract geometry, including algebraic geometry, categorical noncommutative geometry, etc.
- Quantization related to category theory, in particular categorical quantization;
- Categorical physics relevant for mathematics.

In this article and in category theory in general ∞=ω.

Timeline to 1945: before the definitions

Year	Contributors	Event
1890	David Hilbert	Resolution of modules and free resolution of modules.
1890	David Hilbert	Hilbert's syzygy theorem is a prototype for a concept of dimension in homological algebra.
1893	David Hilbert	A fundamental theorem in algebraic geometry, the Hilbert Nullstellensatz. It was later reformulated to: the category of affine varieties over a field k is equivalent to the dual of the category of reduced finitely generated (commutative) k-algebras.
1894	Henri Poincaré	Fundamental group of a topological space.
1895	Henri Poincaré	Simplicial homology.
1895	Henri Poincaré	Fundamental work *Analysis situs*, the beginning of algebraic topology.
c.1910	L. E. J. Brouwer	Brouwer develops intuitionism as a contribution to foundational debate in the period roughly 1910 to 1930 on mathematics, with intuitionistic logic a by-product of an increasingly sterile discussion on formalism.
1923	Hermann Künneth	Künneth formula for homology of product of spaces.
1928	Arend Heyting	Brouwer's intuitionistic logic made into formal mathematics, as logic in which the Heyting algebra replaces the Boolean algebra.
1929	Walther Mayer	Chain complexes.
1930	Ernst Zermelo–Abraham Fraenkel	Statement of the definitive ZF-axioms of set theory, first stated in 1908 and improved upon since then.
c.1930	Emmy Noether	Module theory is developed by Noether and her students, and algebraic topology starts to be properly founded in abstract algebra rather than by *ad hoc* arguments.
1932	Eduard Čech	Čech cohomology.
1933	Solomon Lefschetz	Singular homology of topological spaces.
1934	Reinhold Baer	Ext groups, Ext functor (for abelian groups and with different notation).
1935	Witold Hurewicz	Higher homotopy groups of a topological space.
1936	Marshall Stone	Stone representation theorem for Boolean algebras initiates various Stone dualities.

1937	Richard Brauer–Cecil Nesbitt	Frobenius algebras.
1938	Hassler Whitney	"Modern" definition of cohomology, summarizing the work since James Alexander and Andrey Kolmogorov first defined cochains.
1940	Reinhold Baer	Injective modules.
1940	Kurt Gödel–Paul Bernays	Proper classes in set theory.
1940	Heinz Hopf	Hopf algebras.
1941	Witold Hurewicz	First fundamental theorem of homological algebra: Given a short exact sequence of spaces there exist a connecting homomorphism such that the long sequence of cohomology groups of the spaces is exact.
1942	Samuel Eilenberg–Saunders Mac Lane	Universal coefficient theorem for Čech cohomology; later this became the general universal coefficient theorem. The notations Hom and Ext first appear in their paper.
1943	Norman Steenrod	Homology with local coefficients.
1943	Israel Gelfand–Mark Naimark	Gelfand–Naimark theorem (sometimes called Gelfand isomorphism theorem): The category Haus of locally compact Hausdorff spaces with continuous proper maps as morphisms is equivalent to the category C*Alg of commutative C*-algebras with proper *-homomorphisms as morphisms.
1944	Garrett Birkhoff–Øystein Ore	Galois connections generalizing the Galois correspondence: a pair of adjoint functors between two categories that arise from partially ordered sets (in modern formulation).
1944	Samuel Eilenberg	"Modern" definition of singular homology and singular cohomology.
1945	Beno Eckmann	Defines the cohomology ring building on Heinz Hopf's work.

1945–1970

Year	Contributors	Event
1945	Saunders Mac Lane–Samuel Eilenberg	Start of category theory: axioms for categories, functors and natural transformations.
1945	Norman Steenrod–Samuel Eilenberg	Eilenberg–Steenrod axioms for homology and cohomology.
1945	Jean Leray	Starts sheaf theory: At this time a sheaf was a map assigned a module or a ring to a closed subspace of a topological space. The first example was the sheaf assigning to a closed subspace its p'th cohomology group.
1945	Jean Leray	Defines Sheaf cohomology using his new concept of sheaf.
1946	Jean Leray	Invents spectral sequences as a method for iteratively approximating cohomology groups by previous approximate cohomology groups. In the limiting case it gives the sought cohomology groups.
1948	Cartan seminar	Writes up sheaf theory for the first time.
1948	A. L. Blakers	Crossed complexes (called group systems by Blakers), after a suggestion of Samuel Eilenberg: A nonabelian generalizations of chain complexes of abelian groups which are equivalent to strict ω-groupoids. They form a category Crs that has many satisfactory properties such as a monoidal structure.
1949	John Henry Whitehead	Crossed modules.
1949	André Weil	Formulates the Weil conjectures on remarkable relations between the cohomological structure of algebraic varieties over **C** and the diophantine structure of algebraic varieties over finite fields.

1950	Henri Cartan	In the book Sheaf theory from the Cartan seminar he defines: Sheaf space (étale space), support of sheaves axiomatically, sheaf cohomology with support in an axiomatic form and more.
1950	John Henry Whitehead	Outlines algebraic homotopy program for describing, understanding and calculating homotopy types of spaces and homotopy classes of mappings
1950	Samuel Eilenberg–Joe Zilber	Simplicial sets as a purely algebraic model of well behaved topological spaces. A simplicial set can also be seen as a presheaf on the simplex category. A category is a simplicial set such that the Segal maps are isomorphisms.
1951	Henri Cartan	Modern definition of sheaf theory in which a sheaf is defined using open subsets instead of closed subsets of a topological space and all the open subsets are treated at once. A sheaf on a topological space X becomes a functor reminding of a function defined locally on X, and taking values in sets, abelian groups, commutative rings, modules or generally in any category C. In fact Alexander Grothendieck later made a dictionary between sheaves and functions. Another interpretation of sheaves is as continuously varying sets (a generalization of abstract sets). Its purpose is to provide a unified approach to connect local and global properties of topological spaces and to classify the obstructions for passing from local objects to global objects on a topological space by pasting together the local pieces. The C-valued sheaves on a topological space and their homomorphisms form a category.
1952	William Massey	Invents exact couples for calculating spectral sequences.
1953	Jean-Pierre Serre	Serre C-theory and Serre subcategories.
1955	Jean-Pierre Serre	Shows there is a 1-1 correspondence between algebraic vector bundles over an affine variety and finitely generated projective modules over its coordinate ring (Serre–Swan theorem).
1955	Jean-Pierre Serre	Coherent sheaf cohomology in algebraic geometry.
1956	Jean-Pierre Serre	GAGA correspondence.
1956	Henri Cartan–Samuel Eilenberg	Influential book: *Homological Algebra*, summarizing the state of the art in its topic at that time. The notation Tor_n and Ext^n, as well as the concepts of projective module, projective and injective resolution of a module, derived functor and hyperhomology appear in this book for the first time.
1956	Daniel Kan	Simplicial homotopy theory also called categorical homotopy theory: A homotopy theory completely internal to the category of simplicial sets.
1957	Charles Ehresmann–Jean Bénabou	Pointless topology building on Marshall Stone's work.
1957	Alexander Grothendieck	Abelian categories in homological algebra that combine exactness and linearity.
1957	Alexander Grothendieck	Influential *Tohoku* paper rewrites homological algebra; proving Grothendieck duality (Serre duality for possibly singular algebraic varieties). He also showed that the conceptual basis for homological algebra over a ring also holds for linear objects varying as sheaves over a space.
1957	Alexander Grothendieck	Grothendieck relative point of view, S-schemes.
1957	Alexander Grothendieck	Grothendieck–Hirzebruch–Riemann–Roch theorem for smooth schemes; the proof introduces K-theory.
1957	Daniel Kan	Kan complexes: Simplicial sets (in which every horn has a filler) that are geometric models of simplicial ∞-groupoids. Kan complexes are also the fibrant (and cofibrant) objects of model categories of simplicial sets for which the fibrations are Kan fibrations.
1958	Alexander Grothendieck	Starts new foundation of algebraic geometry by generalizing varieties and other spaces in algebraic geometry to schemes which have the structure of a category with open subsets as objects and restrictions as morphisms. Schemes forma a category that is a Grothendieck topos, and to a scheme and even a stack one may associate a Zariski topos, an étale topos, a fppf topos, a fpqc topos, a Nisnevich topos, a flat topos, ... depending on the topology imposed on the scheme. The whole of algebraic geometry was categorized with time.
1958	Roger Godement	Monads in category theory (then called standard constructions and triples). Monads generalize classical notions from universal algebra and can in this sense be thought of as an algebraic theory over a category: the theory of the category of T-algebras. An algebra for a monad subsumes and generalizes the notion of a model for an algebraic theory.

1958	Daniel Kan	Adjoint functors.
1958	Daniel Kan	Limits in category theory.
1958	Alexander Grothendieck	Fibred categories.
1959	Bernard Dwork	Proves the rationality part of the Weil conjectures (the first conjecture).
1959	Jean-Pierre Serre	Algebraic K-theory launched by explicit analogy of ring theory with geometric cases.
1960	Alexander Grothendieck	Fiber functors
1960	Daniel Kan	Kan extensions
1960	Alexander Grothendieck	Formal algebraic geometry and formal schemes
1960	Alexander Grothendieck	Representable functors
1960	Alexander Grothendieck	Categorizes Galois theory (Grothendieck galois theory)
1960	Alexander Grothendieck	Descent theory: An idea extending the notion of gluing in topology to schemes to get around the brute equivalence relations. It also generalizes localization in topology
1961	Alexander Grothendieck	Local cohomology. Introduced at a seminar in 1961 but the notes are published in 1967
1961	Jim Stasheff	Associahedra later used in the definition of weak n-categories
1961	Richard Swan	Shows there is a 1-1 correspondence between topological vector bundles over a compact Hausdorff space X and finitely generated projective modules over the ring $C(X)$ of continuous functions on X (Serre–Swan theorem)
1963	Frank Adams–Saunders Mac Lane	PROP categories and PACT categories for higher homotopies. PROPs are categories for describing families of operations with any number of inputs and outputs. Operads are special PROPs with operations with only one output
1963	Alexander Grothendieck	Étale topology, a special Grothendieck topology on schemes
1963	Alexander Grothendieck	Étale cohomology
1963	Alexander Grothendieck	Grothendieck toposes, which are categories which are like universes (generalized spaces) of sets in which one can do mathematics
1963	William Lawvere	Algebraic theories and algebraic categories
1963	William Lawvere	Founds Categorical logic, discover internal logics of categories and recognizes its importance and introduces Lawvere theories. Essentially categorical logic is a lift of different logics to being internal logics of categories. Each kind of category with extra structure corresponds to a system of logic with its own inference rules. A Lawvere theory is an algebraic theory as a category with finite products and possessing a "generic algebra" (a generic group). The structures described by a Lawvere theory are models of the Lawvere theory
1963	Jean-Louis Verdier	Triangulated categories and triangulated functors. Derived categories and derived functors are special cases of these
1963	Jim Stasheff	A_∞-algebras: dg-algebra analogs of topological monoids associative up to homotopy appearing in topology (i.e. H-spaces)
1963	Jean Giraud	Giraud characterization theorem characterizing Grothendieck toposes as categories of sheaves over a small site
1963	Charles Ehresmann	Internal category theory: Internalization of categories in a category V with pullbacks is replacing the category Set (same for classes instead of sets) by V in the definition of a category. Internalization is a way to rise the categorical dimension
1963	Charles Ehresmann	Multiple categories and multiple functors

1963	Saunders Mac Lane	Monoidal categories also called tensor categories: Strict 2-categories with one object made by a relabelling trick to categories with a tensor product of objects that is secretly the composition of morphisms in the 2-category. There are several object in a monoidal category since the relabelling trick makes 2-morphisms of the 2-category to morphisms, morphisms of the 2-category to objects and forgets about the single object. In general a higher relabelling trick works for n-categories with one object to make general monoidal categories. The most common examples include: ribbon categories, braided tensor categories, spherical categories, compact closed categories, symmetric tensor categories, modular categories, autonomous categories, categories with duality
1963	Saunders Mac Lane	Mac Lane coherence theorem for determining commutativity of diagrams in monoidal categories
1964	William Lawvere	ETCS Elementary Theory of the Category of Sets: An axiomatization of the category of sets which is also the constant case of an elementary topos
1964	Barry Mitchell–Peter Freyd	Mitchell–Freyd embedding theorem: Every small abelian category admits an exact and full embedding into the category of (left) modules Mod_R over some ring R
1964	Rudolf Haag–Daniel Kastler	Algebraic quantum field theory after ideas of Irving Segal
1964	Alexander Grothendieck	Topologizes categories axiomatically by imposing a Grothendieck topology on categories which are then called sites. The purpose of sites is to define coverings on them so sheaves over sites can be defined. The other "spaces" one can define sheaves for except topological spaces are locales
1964	Michael Artin–Alexander Grothendieck	ℓ-adic cohomology, technical development in SGA4 of the long-anticipated Weil cohomology.
1964	Alexander Grothendieck	Proves the Weil conjectures except the analogue of the Riemann hypothesis
1964	Alexander Grothendieck	Six operations formalism in homological algebra; Rf_*, f^{-1}, $Rf_!$, $f^!$, $\otimes^L$, RHom, and proof of its closedness
1964	Alexander Grothendieck	Introduced in a letter to Jean-Pierre Serre conjectural motives (algebraic geometry) to express the idea that there is a single universal cohomology theory underlying the various cohomology theories for algebraic varieties. According to Grothendiecks philosophy there should be a universal cohomology functor attaching a pure motive h(X) to each smooth projective variety X. When X is not smooth or projective h(X) must be replaced by a more general mixed motive which has a weight filtration whose quotients are pure motivess. The category of motives (the categorical framework for the universal cohomology theory) may be used as an abstract substitute for singular cohomology (and rational cohomology) to compare, relate and unite "motivated" properties and parallel phenomena of the various cohomology theories and to detect topological structure of algebraic varieties. The categories of pure motives and of mixed motives are abelian tensor categories and the category of pure motives is also a Tannakian category. Categories of motives are made by replacing the category of varieties by a category with the same objects but whose morphisms are correspondences, modulo a suitable equivalence relation. Different equivalences give different theories. Rational equivalence gives the category of Chow motives with Chow groups as morphisms which are in some sense universal. Every geometric cohomology theory is a functor on the category of motives. Each induced functor ρ:motives modulo numerical equivalence→graded **Q**-vector spaces is called a realization of the category of motives, the inverse functors are called improvement s. Mixed motives explain phenomena in as diverse areas as: Hodge theory, algebraic K-theory, polylogarithms, regulator maps, automorphic forms, L-functions, ℓ-adic representations, trigonometric sums, homotopy of algebraic varieties, algebraic cycles, moduli spaces and thus has the potential of enriching each area and of unifying them all.
1965	Edgar Brown	Abstract homotopy categories: A proper framework for the study of homotopy theory of CW-complexes
1965	Max Kelly	dg-categories
1965	Max Kelly–Samuel Eilenberg	Enriched category theory: Categories C enriched over a category V are categories with Hom-sets Hom_C not just a set or class but with the structure of objects in the category V. Enrichment over V is a way to rise the categorical dimension
1965	Charles Ehresmann	Defines both strict 2-categories and strict n-categories

1966	Alexander Grothendieck	Crystals (a kind of sheaf used in crystalline cohomology)
1966	William Lawvere	ETAC Elementary theory of abstract categories, first proposed axioms for Cat or category theory using first order logic
1967	Jean Bénabou	Bicategories (weak 2-categories) and weak 2-functors
1967	William Lawvere	Founds synthetic differential geometry
1967	Simon Kochen–Ernst Specker	Kochen–Specker theorem in quantum mechanics
1967	Jean-Louis Verdier	Defines derived categories and redefines derived functors in terms of derived categories
1967	Peter Gabriel–Michel Zisman	Axiomatizes simplicial homotopy theory
1967	Daniel Quillen	Quillen Model categories and Quillen model functors: A framework for doing homotopy theory in an axiomatic way in categories and an abstraction of homotopy categories in such a way that $hC = C[W^{-1}]$ where W^{-1} are the inverted weak equivalences of the Quillen model category C. Quillen model categories are homotopically complete and cocomplete, and come with a built-in Eckmann–Hilton duality
1967	Daniel Quillen	Homotopical algebra (published as a book and also sometimes called noncommutative homological algebra): The study of various model categories and the interplay between fibrations, cofibrations and weak equivalences in arbitrary closed model categories
1967	Daniel Quillen	Quillen axioms for homotopy theory in model categories
1967	Daniel Quillen	First fundamental theorem of simplicial homotopy theory: The category of simplicial sets is a (proper) closed (simplicial) model category
1967	Daniel Quillen	Second fundamental theorem of simplicial homotopy theory: The realization functor and the singular functor is an equivalence of categories hΔ and hTop (Δ the category of simplicial sets)
1967	Jean Bénabou	V-actegories: A category C with an action ⊗ :V × C → C which is associative and unital up to coherent isomorphism, for V a symmetric monoidal category. V-actegories can be seen as the categorification of R-modules over a commutative ring R
1968	Chen Yang-Rodney Baxter	Yang-Baxter equation, later used as a relation in braided monoidal categories for crossings of braids
1968	Alexander Grothendieck	Crystalline cohomology: A p-adic cohomology theory in characteristic p invented to fill the gap left by étale cohomology which is deficient in using mod p coefficients for this case. It is sometimes referred to by Grothendieck as the yoga of de Rham coefficients and Hodge coefficients since crystalline cohomology of a variety X in characteristic p is like de Rham cohomology mod p of X and there is an isomorphism between de Rham cohomology groups and Hodge cohomology groups of harmonic forms
1968	Alexander Grothendieck	Grothendieck connection
1968	Alexander Grothendieck	Formulates the standard conjectures on algebraic cycles
1968	Michael Artin	Algebraic spaces in algebraic geometry as a generalization of schemes
1968	Charles Ehresmann	Sketches (category theory): An alternative way of presenting a theory (which is categorical in character as opposed to linguistic) whose models are to study in appropriate categories. A sketch is a small category with a set of distinguished cones and a set of distinguished cocones satisfying some axioms. A model of a sketch is a set-valued functor transforming the distinguished cones into limit cones and the distinguished cocones into colimit cones. The categories of models of sketches are exactly the accessible categories
1968	Joachim Lambek	Multicategories

1969	Max Kelly-Nobuo Yoneda	Ends and coends
1969	Pierre Deligne-David Mumford	Deligne-Mumford stacks as a generalization of schemes
1969	William Lawvere	Doctrines (category theory), a doctrine is a monad on a 2-category
1970	William Lawvere-Myles Tierney	Elementary toposes: Categories modeled after the category of sets which are like universes (generalized spaces) of sets in which one can do mathematics. One of many ways to define a topos is: a properly cartesian closed category with a subobject classifier. Every Grothendieck topos is an elementary topos
1970	John Conway	Skein theory of knots: The computation of knot invariants by skein modules. Skein modules can be based on quantum invariants

1971–1980

Year	Contributors	Event
1971	Saunders Mac Lane	Influential book: Categories for the working mathematician, which became the standard reference in category theory
1971	Horst Herrlich-Oswald Wyler	Categorical topology: The study of topological categories of structured sets (generalizations of topological spaces, uniform spaces and the various other spaces in topology) and relations between them, culminating in universal topology. General categorical topology study and uses structured sets in a topological category as general topology study and uses topological spaces. Algebraic categorical topology tries to apply the machinery of algebraic topology for topological spaces to structured sets in a topological category.
1971	Harold Temperley-Elliott Lieb	Temperley–Lieb algebras: Algebras of tangles defined by generators of tangles and relations among them
1971	William Lawvere–Myles Tierney	Lawvere–Tierney topology on a topos
1971	William Lawvere–Myles Tierney	Topos theoretic forcing (forcing in toposes): Categorization of the set theoretic forcing method to toposes for attempts to prove or disprove the continuum hypothesis, independence of the axiom of choice, etc. in toposes
1971	Bob Walters-Ross Street	Yoneda structures on 2-categories
1971	Roger Penrose	String diagrams to manipulate morphisms in a monoidal category
1971	Jean Giraud	Gerbes: Categorified principal bundles that are also special cases of stacks
1971	Joachim Lambek	Generalizes the Haskell-Curry-William-Howard correspondence to a three way isomorphism between types, propositions and objects of a cartesian closed category
1972	Max Kelly	Clubs (category theory) and coherence (category theory). A club is a special kind of 2-dimensional theory or a monoid in Cat/(category of finite sets and permutations P), each club giving a 2-monad on Cat
1972	John Isbell	Locales: A "generalized topological space" or "pointless spaces" defined by a lattice (a complete Heyting algebra also called a Brouwer lattice) just as for a topological space the open subsets form a lattice. If the lattice possess enough points it is a topological space. Locales are the main objects of pointless topology, the dual objects being frames. Both locales and frames form categories that are each others opposite. Sheaves can be defined over locales. The other "spaces" one can define sheaves over are sites. Although locales were known earlier John Isbell first named them
1972	Ross Street	Formal theory of monads: The theory of monads in 2-categories
1972	Peter Freyd	Fundamental theorem of topos theory: Every slice category (E,Y) of a topos E is a topos and the functor f*:(E,X)→(E,Y) preserves exponentials and the subobject classifier object Ω and has a right and left adjoint functor
1972	Alexander Grothendieck	Universes (mathematics) for sets

1972	Jean Bénabou–Ross Street	Cosmoses (category theory) which categorize universes: A cosmos is a generalized universe of 1-categories in which you can do category theory. When set theory is generalized to the study of a Grothendieck topos, the analogous generalization of category theory is the study of a cosmos. Ross Street definition: A bicategory such that 1) small bicoproducts exist 2) each monad admits a Kleisli construction (analogous to the quotient of an equivalence relation in a topos) 3) it is locally small-cocomplete 4) there exists a small Cauchy generator. Cosmoses are closed under dualization, parametrization and localization. Ross Street also introduces elementary cosmoses. Jean Bénabou definition: A bicomplete symmetric monoidal closed category
1972	Peter May	Operads: An abstraction of the family of composable functions of several variables together with an action of permutation of variables. Operads can be seen as algebraic theories and algebras over operads are then models of the theories. Each operad gives a monad on Top. Multicategories with one object are operads. PROPs generalize operads to admit operations with several inputs and several outputs. Operads are used in defining opetopes, higher category theory, homotopy theory, homological algebra, algebraic geometry, string theory and many other areas.
1972	William Mitchell-Jean Bénabou	Mitchell-Bénabou internal language of a toposes: For a topos E with subobject classifier object Ω a language (or type theory) L(E) where: 1) the types are the objects of E 2) terms of type X in the variables x_i of type X_i are polynomial expressions $\varphi(x_1,...,x_m)$:1→X in the arrows x_i:1→X_i in E 3) formulas are terms of type Ω (arrows from types to Ω) 4) connectives are induced from the internal Heyting algebra structure of Ω 5) quantifiers bounded by types and applied to formulas are also treated 6) for each type X there are also two binary relations $=_X$ (defined applying the diagonal map to the product term of the arguments) and $\in_X$ (defined applying the evaluation map to the product of the term and the power term of the arguments). A formula is true if the arrow which interprets it factor through the arrow true:1→Ω. The Mitchell-Bénabou internal language is a powerful way to describe various objects in a topos as if they were sets and hence is a way of making the topos into a generalized set theory, to write and prove statements in a topos using first order intuitionistic predicate logic, to consider toposes as type theories and to express properties of a topos. Any language L also generates a linguistic topos E(L)
1973	Chris Reedy	Reedy categories: Categories of "shapes" that can be used to do homotopy theory. A Reedy category is a category R equipped with a structure enabling the inductive construction of diagrams and natural transformations of shape R. The most important consequence of a Reedy structure on R is the existence of a model structure on the functor category M^R whenever M is a model category. Another advantage of the Reedy structure is that its cofibrations, fibrations and factorizations are explicit. In a Reedy category there is a notion of an injective and a surjective morphism such that any morphism can be factored uniquely as a surjection followed by an injection. Examples are the ordinal α considered as a poset and hence a category. The opposite R° of a Reedy category R is a Reedy category. The simplex category Δ and more generally for any simplicial set X its category of simplices Δ/X is a Reedy category. The model structure on M^Δ for a model category M is described in an unpublished manuscript by Chris Reedy
1973	Kenneth Brown–Stephen Gersten	Shows the existence of a global closed model structure on the categegory of simplicial sheaves on a topological space, with weak assumptions on the topological space
1973	Kenneth Brown	Generalized sheaf cohomology of a topological space X with coefficients a sheaf on X with values in Kans category of spectra with some finiteness conditions. It generalizes generalized cohomology theory and sheaf cohomology with coefficients in a complex of abelian sheaves
1973	William Lawvere	Finds that Cauchy completeness can be expressed for general enriched categories with the category of generalized metric spaces as a special case. Cauchy sequences become left adjoint modules and convergence become representability
1973	Jean Bénabou	Distributors (also called modules, profunctors, directed bridges)
1973	Pierre Deligne	Proves the last of the Weil conjectures, the analogue of the Riemann hypothesis

1973	John Boardman-Rainer Vogt	Segal categories: Simplicial analogues of A_∞-categories. They naturally generalize simplicial categories, in that they can be regarded as simplicial categories with composition only given up to homotopy. Def: A simplicial space X such that X_0 (the set of points) is a discrete simplicial set and the Segal map $\varphi_k : X_k \to X_1 \times_X 0 \ldots \times_X 0X_1$ (induced by $X(\alpha_i):X_k \to X_1$) assigned to X is a weak equivalence of simplicial sets for k≥2. Segal categories are a weak form of S-categories, in which composition is only defined up to a coherent system of equivalences. Segal categories were defined one year later implicitly by Graeme Segal. They were named Segal categories first by William Dwyer–Daniel Kan–Jeffrey Smith 1989. In their famous book Homotopy invariant algebraic structures on topological spaces John Boardman and Rainer Vogt called them quasi-categories. A quasi-category is a simplicial set satisfying the weak Kan condition, so quasi-categories are also called weak Kan complexes
1973	Daniel Quillen	Frobenius categories: An exact category in which the classes of injective and projective objects coincide and for all objects x in the category there is a deflation P(x)→x (the projective cover of x) and an inflation x→I(x) (the injective hull of x) such that both P(x) and I(x) are in the category of pro/injective objects. A Frobenius category E is an example of a model category and the quotient E/P (P is the class of projective/injective objects) is its homotopy category hE
1974	Michael Artin	Generalizes Deligne–Mumford stacks to Artin stacks
1974	Robert Paré	Paré monadicity theorem: E is a topos→E° is monadic over E
1974	Andy Magid	Generalizes Grothendiecks Galois theory from groups to the case of rings using Galois groupoids
1974	Jean Bénabou	Logic of fibred categories
1974	John Gray	Gray categories with Gray tensor product
1974	Kenneth Brown	Writes a very influential paper that defines Browns categories of fibrant objects and dually Brown categories of cofibrant objects
1974	Shiing-Shen Chern–James Simons	Chern–Simons theory: A particular TQFT which describe knot and manifold invariants, at that time only in 3D
1975	Saul Kripke–André Joyal	Kripke–Joyal semantics of the Mitchell–Bénabou internal language for toposes: The logic in categories of sheaves is first order intuitionistic predicate logic
1975	Radu Diaconescu	Diaconescu theorem: The internal axiom of choice holds in a topos → the topos is a boolean topos. So in IZF the axiom of choice implies the law of excluded middle
1975	Manfred Szabo	Polycategories
1975	William Lawvere	Observes that Delignes theorem about enough points in a coherent topos implies the Gödel completeness theorem for first order logic in that topos
1976	Alexander Grothendieck	Schematic homotopy types
1976	Marcel Crabbe	Heyting categories also called logoses: Regular categories in which the subobjects of an object form a lattice, and in which each inverse image map has a right adjoint. More precisely a coherent category C such that for all morphisms f:A→B in C the functor $f^*:Sub_C(B)\to Sub_C(A)$ has a left adjoint and a right adjoint. $Sub_C(A)$ is the preorder of subobjects of A (the full subcategory of C/A whose objects are subobjects of A) in C. Every topos is a logos. Heyting categories generalize Heyting algebras.
1976	Ross Street	Computads
1977	Michael Makkai–Gonzalo Reyes	Develops the Mitchell–Bénabou internal language of a topos thoroughly in a more general setting
1977	Andre Boileau–André Joyal–Jon Zangwill	LST Local set theory: Local set theory is a typed set theory whose underlying logic is higher order intuitionistic logic. It is a generalization of classical set theory, in which sets are replaced by terms of certain types. The category C(S) built out of a local theory S whose objects are the local sets (or S-sets) and whose arrows are the local maps (or S-maps) is a linguistic topos. Every topos E is equivalent to a linguistic topos C(S(E))

1977	John Roberts	Introduces most general nonabelian cohomology of ω-categories with ω-categories as coefficients when he realized that general cohomology is about coloring simplices in ω-categories. There are two methods of constructing general nonabelian cohomology, as nonabelian sheaf cohomology in terms of descent for ω-category valued sheaves, and in terms of homotopical cohomology theory which realizes the cocycles. The two approaches are related by codescent
1978	John Roberts	Complicial sets (simplicial sets with structure or enchantment)
1978	Francois Bayen–Moshe Flato–Chris Fronsdal–Andre Lichnerowicz–Daniel Sternheimer	Deformation quantization, later to be a part of categorical quantization
1978	André Joyal	Combinatorial species in enumerative combinatorics
1978	Don Anderson	Building on work of Kenneth Brown defines ABC (co)fibration categories for doing homotopy theory and more general ABC model categories, but the theory lies dormant until 2003. Every Quillen model category is an ABC model category. A difference to Quillen model categories is that in ABC model categories fibrations and cofibrations are independent and that for an ABC model category M^D is an ABC model category. To a ABC (co)fibration category is canonically associated a (left) right Heller derivator. Topological spaces with homotopy equivalences as weak equivalences, Hurewicz cofibrations as cofibrations and Hurewicz fibrations as fibrations form an ABC model category, the Hurewicz model structure on Top. Complexes of objects in an abelian category with quasi-isomorphisms as weak equivalences and monomorphisms as cofibrations form an ABC precofibration category
1979	Don Anderson	Anderson axioms for homotopy theory in categories with a fraction functor
1980	Alexander Zamolodchikov	Zamolodchikov equation also called tetrahedron equation
1980	Ross Street	Bicategorical Yoneda lemma
1980	Masaki Kashiwara–Zoghman Mebkhout	Proves the Riemann–Hilbert correspondence for complex manifolds
1980	Peter Freyd	Numerals in a topos

1981–1990

Year	Contributors	Event
1981	Shigeru Mukai	Mukai–Fourier transform
1982	Bob Walters	Enriched categories with bicategories as a base
1983	Alexander Grothendieck	Pursuing stacks: Correspondence by mail with Daniel Quillen about Alexander Grothendiecks mathematical visions written down in a 629 pages manuscript
1983	Alexander Grothendieck	First appearance of strict ∞-categories in pursuing stacks
1983	Alexander Grothendieck	Fundamental infinity groupoid: A complete homotopy invariant $\Pi_\infty(X)$ for CW-complexes X. The inverse functor is the geometric realization functor |.| and together they form an "equivalence" between the category of CW-complexes and the category of ω-groupoids
1983	Alexander Grothendieck	Homotopy hypothesis: The homotopy category of CW-complexes is Quillen equivalent to a homotopy category of reasonable weak ∞-groupoids
1983	Alexander Grothendieck	Grothendieck derivators: A model for homotopy theory similar to Quilen model categories but more satisfactory. Grothendieck derivators are dual to Heller derivators
1983	Alexander Grothendieck	Elementary modelizers: Categories of presheaves that modelize homotopy types (thus generalizing the theory of simplicial sets). Canonical modelizers are also used in pursuing stacks

1983	Alexander Grothendieck	Smooth functors and proper functors
1984	Vladimir Bazhanov–Razumov Stroganov	Bazhanov–Stroganov d-simplex equation generalizing the Yang–Baxter equation and the Zamolodchikov equation
1984	Horst Herrlich	Universal topology in categorical topology: A unifying categorical approach to the different structured sets (topological structures such as topological spaces and uniform spaces) whose class form a topological category similar as universal algebra is for algebraic structures
1984	André Joyal	Simplicial sheaves (sheaves with values in simplicial sets). Simplicial sheaves on a topological space X is a model for the hypercomplete ∞-topos $Sh(X)^{\wedge}$
1984	André Joyal	Shows that the category of simplicial objects in a Grothendieck topos has a closed model structure
1984	André Joyal–Myles Tierney	Main Galois theorem for toposes: Every topos is equivalent to a category of étale presheaves on an open étale groupoid
1985	Michael Schlessinger–Jim Stasheff	L_{∞}-algebras
1985	André Joyal–Ross Street	Braided monoidal categories
1985	André Joyal–Ross Street	Joyal–Street coherence theorem for braided monoidal categories
1985	Paul Ghez–Ricardo Lima–John Roberts	C*-categories
1986	Joachim Lambek–Phil Scott	Influential book: Introduction to higher order categorical logic
1986	Joachim Lambek–Phil Scott	Fundamental theorem of topology: The section-functor Γ and the germ-functor Λ establish a dual adjunction between the category of presheaves and the category of bundles (over the same topological space) which restricts to a dual equivalence of categories (or duality) between corresponding full subcategories of sheaves and of étale bundles
1986	Peter Freyd–David Yetter	Constructs the (compact braided) monoidal category of tangles
1986	Vladimir Drinfel'd–Michio Jimbo	Quantum groups: In other words quasitriangular Hopf algebras. The point is that the categories of representations of quantum groups are tensor categories with extra structure. They are used in construction of quantum invariants of knots and links and low dimensional manifolds, representation theory, q-deformation theory, CFT, integrable systems. The invariants are constructed from braided monoidal categories that are categories of representations of quantum groups. The underlying structure of a TQFT is a modular category of representations of a quantum group
1986	Saunders Mac Lane	Mathematics, form and function (a foundation of mathematics)
1987	Jean-Yves Girard	Linear logic: The internal logic of a linear category (an enriched category with its Hom-sets being linear spaces)
1987	Peter Freyd	Freyd representation theorem for Grothendieck toposes
1987	Ross Street	Definition of the nerve of a weak n-category and thus obtaining the first definition of weak "n"-category using simplices
1987	Ross Street–John Roberts	Formulates Street–Roberts conjecture: Strict ω-categories are equivalent to complicial sets
1987	André Joyal–Ross Street–Mei Chee Shum	Ribbon categories: A balanced rigid braided monoidal category
1987	Ross Street	n-computads
1987	Iain Aitchison	Bottom up Pascal triangle algorithm for computing nonabelian n-cocycle conditions for nonabelian cohomology
1987	Vladimir Drinfel'd-Gérard Laumon	Formulates geometric Langlands program

1987	Vladimir Turaev	Starts quantum topology by using quantum groups and R-matrices to giving an algebraic unification of most of the known knot polynomials. Especially important was Vaughan Jones and Edward Wittens work on the Jones polynomial
1988	Alex Heller	Heller axioms for homotopy theory as a special abstract hyperfunctor. A feature of this approach is a very general localization
1988	Alex Heller	Heller derivators, the dual of Grothendieck derivators
1988	Alex Heller	Gives a global closed model structure on the category of simplicial presheaves. John Jardine has also given a model structure for the category of simplicial presheaves
1988	Graeme Segal	Elliptic objects: A functor that is a categorified version of a vector bundle equipped with a connection, it is a 2D parallel transport for strings
1988	Graeme Segal	Conformal field theory CFT: A symmetric monoidal functor $Z:nCob_C \to Hilb$ satisfying some axioms
1988	Edward Witten	Topological quantum field theory TQFT: A monoidal functor $Z:nCob \to Hilb$ satisfying some axioms
1988	Edward Witten	Topological string theory
1989	Hans Baues	Influential book: Algebraic homotopy
1989	Michael Makkai-Robert Paré	Accessible categories: Categories with a "good" set of generators allowing to manipulate large categories as if they were small categories, without the fear of encountering any set-theoretic paradoxes. Locally presentable categories are complete accessible categories. Accessible categories are the categories of models of sketches. The name comes from that these categories are accessible as models of sketches.
1989	Edward Witten	Witten functional integral formalism and Witten invariants for manifolds.
1990	Peter Freyd	Allegories (category theory): An abstraction of the category of sets and relations as morphisms, it bears the same resemblance to binary relations as categories do to functions and sets. It is a category in which one has in addition to composition a unary operation reciprocation R° and a partial binary operation intersection $R \cap S$, like in the category of sets with relations as morphisms (instead of functions) for which a number of axioms are required. It generalizes the relation algebra to relations between different sorts.
1990	Nicolai Reshetikhin–Vladimir Turaev–Edward Witten	Reshetikhin–Turaev–Witten invariants of knots from modular tensor categories of representations of quantum groups.

1991–2000

Year	Contributors	Event
1991	Jean-Yves Girard	Polarization of linear logic.
1991	Ross Street	Parity complexes. A parity complex generates a free ω-category.
1991	André Joyal-Ross Street	Formalization of Penrose string diagrams to calculate with abstract tensors in various monoidal categories with extra structure. The calculus now depends on the connection with low dimensional topology.
1991	Ross Street	Definition of the descent strict ω-category of a cosimplicial strict ω-category.
1991	Ross Street	Top down excision of extremals algorithm for computing nonabelian n-cocycle conditions for nonabelian cohomology.
1992	Yves Diers	Axiomatic categorical geometry using algebraic-geometric categories and algebraic-geometric functors.
1992	Saunders Mac Lane-Ieke Moerdijk	Influential book: *Sheaves in geometry and logic*.
1992	John Greenlees-Peter May	Greenlees-May duality

1992	Vladimir Turaev	Modular tensor categories. Special tensor categories that arise in constructiong knot invariants, in constructing TQFTs and CFTs, as truncation (semisimple quotient) of the category of representations of a quantum group (at roots of unity), as categories of representations of weak Hopf algebras, as category of representations of a RCFT.
1992	Vladimir Turaev-Oleg Viro	Turaev-Viro state sum models based on spherical categories (the first state sum models) and Turaev-Viro state sum invariants for 3-manifolds.
1992	Vladimir Turaev	Shadow world of links: Shadows of links give shadow invariants of links by shadow state sums.
1993	Ruth Lawrence	Extended TQFTs
1993	David Yetter-Louis Crane	Crane-Yetter state sum models based on ribbon categories and Crane-Yetter state sum invariants for 4-manifolds.
1993	Kenji Fukaya	A_∞-categories and A_∞-functors: Most commonly in homological algebra, a category with several compositions such that the first composition is associative up to homotopy which satisfies an equation that holds up to another homotopy, etc. (associative up to higher homotopy). A stands for associative. Def: A category C such that 1) for all X,Y in Ob(C) the Hom-sets $\mathrm{Hom}_C(X,Y)$ are finite dimensional chain complexes of **Z**-graded modules 2) for all objects $X_1,...,X_n$ in Ob(C) there is a family of linear composition maps (the higher compositions) $m_n : \mathrm{Hom}_C(X_0,X_1) \otimes \mathrm{Hom}_C(X_1,X_2) \otimes ... \otimes \mathrm{Hom}_C(X_{n-1},X_n) \to \mathrm{Hom}_C(X_0,X_n)$ of degree n-2 (homological grading convention is used) for $n\geq1$ 3) m_1 is the differential on the chain complex $\mathrm{Hom}_C(X,Y)$ 4) m_n satisfy the quadratic A_∞-associativity equation for all $n\geq0$. m_1 and m_2 will be chain maps but the compositions m_i of higher order are not chain maps, nevertheless they are Massey products. In particular it is a linear category. Examples are the Fukaya category Fuk(X) and loop space ΩX where X is a topological space and A_∞-algebras as A_∞-categories with one object. When there are no higher maps (trivial homotopies) C is a dg-category. Every A_∞-category is quasiisomorphic in a functorial way to a dg-category. A quasiisomorphism is a chain map that is an isomorphism in homology. The framework of dg-categories and dg-functors is too narrow for many problems, and it is preferable to consider the wider class of A_∞-categories and A_∞-functors. Many features of A_∞-categories and A_∞-functors come from the fact that they form a symmetric closed multicategory, which is revealed in the language of comonads. From a higher dimensional perspective A_∞-categories are weak ω-categories with all morphisms invertible. A_∞-categories can also be viewed as noncommutative formal dg-manifolds with a closed marked subscheme of objects.
1993	John Barret-Bruce Westbury	Spherical categories: Monoidal categories with duals for diagrams on spheres instead for in the plane.
1993	Maxim Kontsevich	Kontsevich invariants for knots (are perturbation expansion Feynman integrals for the Witten functional integral) defined by the Kontsevich integral. They are the universal Vassiliev invariants for knots.
1993	Daniel Freed	A new view on TQFT using modular tensor categories that unifies 3 approaches to TQFT (modular tensor categories from path integrals).
1994	Francis Borceux	*Handbook of categorical algebra* (3 volumes).
1994	Jean Bénabou-Bruno Loiseau	Orbitals in a topos.
1994	Maxim Kontsevich	Formulates homological mirror symmetry conjecture: X a compact symplectic manifold with first chern class $c_1(X)=0$ and Y a compact Calabi–Yau manifold are mirror pairs if and only if $D(\mathrm{Fuk}_X)$ (the derived category of the Fukaya triangulated category of X concocted out of Lagrangian cycles with local systems) is equivalent to a subcategory of $D^b(\mathrm{Coh}_Y)$ (the bounded derived category of coherent sheaves on Y).
1994	Louis Crane-Igor Frenkel	Hopf categories and construction of 4D TQFTs by them.
1994	John Fischer	Defines the 2-category of 2-knots (knotted surfaces).
1995	Bob Gordon-John Power-Ross Street	Tricategories and a corresponding coherence theorem: Every weak 3-category is equivalent to a Gray 3-category.
1995	Ross Street-Dominic Verity	Surface diagrams for tricategories.
1995	Louis Crane	Coins categorification leading to the categorical ladder.

1995	Sjoerd Crans	A general procedure of transferring closed model structures on a category along adjoint functor pairs to another category.
1995	André Joyal-Ieke Moerdijk	AST Algebraic set theory: Also sometimes called categorical set theory started to develop in 1988 by André Joyal and Ieke Moerdijk and was first presented in detail as a book in 1995 by them. AST is a robust framework based on category theory to study and organize set theories and to construct models of set theories. The aim of AST is to provide a uniform categorical semantics or description of set theories of different kinds (classical or constructive, bounded, predicative or impredicative, well founded or non well founded,...), the various constructions of the cumulative hierarchy of sets, forcing models, sheaf models and realisability models. Instead of focusing on categories of sets AST focuses on categories of classes. The basic tool of AST is the notion of a category with class structure (a category of classes equipped with a class of small maps (the intuition being that their fibres are small in some sense), powerclasses and a universal object (a universe)) which provides an axiomatic framework in which models of set theory can be constructed. The notion of a class category permits both the definition of ZF-algebras (Zermelo-Fraenkel algebra) and related structures expressing the idea that the hierarchy of sets is an algebraic structure on the onehand and the interpretation of the first order logic of elementary set theory on the other. The subcategory of sets in a class category is an elementary topos and every elementary topos occurs as sets in a class category. The class category itself always embeds into the ideal completion of a topos. The interpretation of the logic is that in every class category the universe is a model of basic intuitionistic set theory BIST that is logically complete with respect to class category models. Therefore class categories generalize both topos theory and intuitionistic set theory. AST founds and formalizes set theory on the ZF-algebra with operations union and successor (singleton) instead of on the membership relation. The ZF-axioms are nothing but a description of the free ZF-algebra just as the Peano axioms are a description of the free monoid on one generator. In this perspective the models of set theory are algebras for a suitably presented algebraic theory and many familiar set theoretic conditions (such as well foundedness) are related to familiar algebraic conditions (such as freeness). Using an auxiliary notion of small map it is possible to extend the axioms of a topos and provide a general theory for uniformly constructing models of set theory out of toposes.
1995	Michael Makkai	SFAM Structuralist foundation of abstract mathematics. In SFAM the universe consists of higher dimensional categories, functors are replaced by saturated anafunctors, sets are abstract sets, the formal logic for entities is FOLDS (first-order logic with dependent sorts) in which the identity relation is not given a priori by first order axioms but derived from within a context.
1995	John Baez-James Dolan	Opetopic sets (opetopes) based on operads. Weak n-categories are n-opetopic sets.
1995	John Baez-James Dolan	Introduces the periodic table of mathematics which identifies k-tuply monoidal n-categories. It mirror the table of homotopy groups of the spheres.
1995	John Baez-James Dolan	Outline a program in which n-dimensional TQFTs are described as n-category representations.
1995	John Baez-James Dolan	Proposes n-dimensional deformation quantization.
1995	John Baez-James Dolan	Tangle hypothesis: The n-category of framed n-tangles in n+k dimensions is (n+k)-equivalent to the free weak k-tuply monoidal n-category with duals on one object.
1995	John Baez-James Dolan	Cobordism hypothesis (Extended TQFT hypothesis I): The n-category of which n-dimensional extended TQFTs are representations nCob is the free stable weak n-category with duals on one object.
1995	John Baez-James Dolan	Stabilization hypothesis: After suspending a weak n-category n+2 times, further suspensions have no essential effect. The suspension functor $S:nCat_k \to nCat_{k+1}$ is an equivalence of categories for k=n+2.
1995	John Baez-James Dolan	Extended TQFT hypothesis II: An n-dimensional unitary extended TQFT is a weak n-functor, preserving all levels of duality, from the free stable weak n-category with duals on one object to nHilb.
1995	Valentin Lychagin	Categorical quantization
1995	Pierre Deligne-Vladimir Drinfel'd-Maxim Kontsevich	Derived algebraic geometry with derived schemes and derived moduli stacks. A program of doing algebraic geometry and especially moduli problems in the derived category of schemes or algebraic varieties instead of in their normal categories.
1997	Maxim Kontsevich	Formal deformation quantization theorem: Every Poisson manifold admits a differentiable star product and they are classified up to equivalence by formal deformations of the Poisson structure.

1998	Claudio Hermida-Michael-Makkai-John Power	Multitopes, Multitopic sets.
1998	Carlos Simpson	Simpson conjecture: Every weak ∞-category is equivalent to a ∞-category in which composition and exchange laws are strict and only the unit laws are allowed to hold weakly. It is proven for 1,2,3-categories with a single object.
1998	André Hirschowitz-Carlos Simpson	Give a model category structure on the category of Segal categories. Segal categories are the fibrant-cofibrant objects and Segal maps are the weak equivalences. In fact they generalize the definition to that of a Segal n-category and give a model structure for Segal n-categories for any $n \geq 1$.
1998	Chris Isham-Jeremy Butterfield	Kochen-Specker theorem in topos theory of presheaves: The spectral presheaf (the presheaf that assigns to each operator its spectrum) has no global elements (global sections) but may have partial elements or local elements. A global element is the analogue for presheaves of the ordinary idea of an element of a set. This is equivalent in quantum theory to the spectrum of the C*-algebra of observables in a topos having no points.
1998	Richard Thomas	Richard Thomas, a student of Simon Donaldson, introduces Donaldson–Thomas invariants which are systems of numerical invariants of complex oriented 3-manifolds X, analogous to Donaldson invariants in the theory of 4-manifolds. They are certain weighted Euler characteristics of the moduli space of sheaves on *X* and "count" Gieseker semistable coherent sheaves with fixed Chern character on X. Ideally the moduli spaces should be a critical sets of holomorphic Chern–Simons functions and the Donaldson–Thomas invariants should be the number of critical points of this function, counted correctly. Currently such holomorphic Chern–Simons functions exist at best locally.
1998	John Baez	Spin foam models: A 2-dimensional cell complex with faces labeled by representations and edges labeled by intertwining operators. Spin foams are functors between spin network categories. Any slice of a spin foam gives a spin network.
1998	John Baez–James Dolan	Microcosm principle: Certain algebraic structures can be defined in any category equipped with a categorified version of the same structure.
1998	Alexander Rosenberg	Noncommutative schemes: The pair $(\mathrm{Spec}(A),O_A)$ where A is an abelian category and to it is associated a topological space Spec(A) together with a sheaf of rings O_A on it. In the case when A = QCoh(X) for X a scheme the pair $(\mathrm{Spec}(A),O_A)$ is naturally isomorphic to the scheme (X^{Zar},O_X) using the equivalence of categories $\mathrm{QCoh}(\mathrm{Spec}(R))=\mathrm{Mod}_R$. More generally abelian categories or triangulated categories or dg-categories or A_∞-categories should be regarded as categories of quasicoherent sheaves (or complexes of sheaves) on noncommutative schemes. This is a starting point in noncommutative algebraic geometry. It means that one can think of the category A itself as a space. Since A is abelian it allows to naturally do homological algebra on noncommutative schemes and hence sheaf cohomology.
1998	Maxim Kontsevich	Calabi–Yau categories: A linear category with a trace map for each object of the category and an associated symmetric (with respects to objects) nondegenerate pairing to the trace map. If X is a smooth projective Calabi–Yau variety of dimension d then $D^b(\mathrm{Coh}(X))$ is a unital Calabi–Yau A_∞-category of Calabi–Yau dimension d. A Calabi–Yau category with one object is a Frobenius algebra.
1999	Joseph Bernstein–Igor Frenkel–Mikhail Khovanov	Temperley–Lieb categories: Objects are enumerated by nonnegative integers. The set of homomorphisms from object n to object m is a free R-module with a basis over a ring R. R is given by the isotopy classes of systems of $(\|n\| + \|m\|)/2$ simple pairwise disjoint arcs inside a horizontal strip on the plane that connect in pairs $\|n\|$ points on the bottom and $\|m\|$ points on the top in some order. Morphisms are composed by concatenating their diagrams. Temperley–Lieb categories are categorized Temperley–Lieb algebras.
1999	Moira Chas–Dennis Sullivan	Constructs String topology by cohomology. This is string theory on general topological manifolds.
1999	Mikhail Khovanov	Khovanov homology: A homology theory for knots such that the dimensions of the homology groups are the coefficients of the Jones polynomial of the knot.
1999	Vladimir Turaev	Homotopy quantum field theory HQFT
1999	Vladimir Voevodsky–Fabien Morel	Constructs the homotopy category of schemes.
1999	Ronald Brown–George Janelidze	2-dimensional Galois theory

2000	Vladimir Voevodsky	Gives two constructions of motivic cohomology of varieties, by model categories in homotopy theory and by a triangulated category of DM-motives.
2000	Yasha Eliashberg–Alexander Givental–Helmut Hofer	Symplectic field theory SFT: A functor Z from a geometric category of framed Hamiltonian structures and framed cobordisms between them to an algebraic category of certain differential D-modules and Fourier integral operators between them and satisfying some axioms.
2000	Paul Taylor[1]	ASD (Abstract Stone duality): A reaxiomatisation of the space and maps in general topology in terms of λ-calculus of computable continuous functions and predicates that is both constructive and computable. The topology on a space is treated not as a lattice, but as an exponential object of the same category as the original space, with an associated λ-calculus. Every expression in the λ-calculus denotes both a continuous function and a program. ASD does not use the category of sets, but the full subcategory of overt discrete objects plays this role (an overt object is the dual to a compact object), forming an arithmetic universe (pretopos with lists) with general recursion.

2001–present

Year	Contributors	Event
2001	Charles Rezk	Constructs a model category with certain generalized Segal categories as the fibrant objects, thus obtaining a model for a homotopy theory of homotopy theories. Complete Segal spaces are introduced at the same time.
2001	Charles Rezk	Model toposes and their generalization homotopy toposes (a model topos without the t-completness assumption).
2002	Bertrand Toën-Gabriele Vezzosi	Segal toposes coming from Segal topologies, Segal sites and stacks over them.
2002	Bertrand Toën-Gabriele Vezzosi	Homotopical algebraic geometry: The main idea is to extend schemes by formally replacing the rings with any kind of "homotopy-ring-like object". More precisely this object is a commutative monoid in a symmetric monoidal category endowed with a notion of equivalences which are understood as "up-to-homotopy monoid" (e.g. E_∞-rings).
2002	Peter Johnstone	Influential book: sketches of an elephant - a topos theory compendium. It serves as an encyclopedia of topos theory (2/3 volumes published as of 2008).
2002	Dennis Gaitsgory-Kari Vilonen-Edward Frenkel	Proves the geometric Langlands program for GL(n) over finite fields.
2003	Denis-Charles Cisinski	Makes further work on ABC model categories and brings them back into light. From then they are called ABC model categories after their contributors.
2004	Dennis Gaitsgory	Extended the proof of the geometric Langlands program to include GL(n) over **C**. This allows to consider curves over **C** instead of over finite fields in the geometric Langlands program.
2004	Mario Caccamo	Formal category theoretical expanded λ-calculus for categories.
2004	Francis Borceux-Dominique Bourn	Homological categories
2004	William Dwyer-Philips Hirschhorn-Daniel Kan-Jeffrey Smith	Introduces in the book: Homotopy limit functors on model categories and homotopical categories, a formalism of homotopical categories and homotopical functors (weak equivalence preserving functors) that generalize the model category formalism of Daniel Quillen. A homotopical category has only a distinguished class of morphisms (containing all isomorphisms) called weak equivalences and satisfy the two out of six axiom. This allow to define homotopical versions of initial and terminal objects, limit and colimit functors (that are computed by local constructions in the book), completeness and cocompleteness, adjunctions, Kan extensions and universal properties.
2004	Dominic Verity	Proves the Street-Roberts conjecture.
2004	Ross Street	Definition of the descent weak ω-category of a cosimplicial weak ω-category.

2004	Ross Street	Characterization theorem for cosmoses: A bicategory M is a cosmos iff there exists a base bicategory W such that M is biequivalent to Mod_W. W can be taken to be any full subbicategory of M whose objects form a small Cauchy generator.
2004	Ross Street-Brian Day	Quantum categories and quantum groupoids: A quantum category over a braided monoidal category V is an object R with an opmorphism $h:R^{op} \otimes R \rightarrow A$ into a pseudomonoid A such that h^* is strong monoidal (preserves tensor product and unit up to coherent natural isomorphisms) and all R, h and A lie in the autonomous monoidal bicategory $Comod(V)^{co}$ of comonoids. $Comod(V)=Mod(V^{op})^{coop}$. Quantum categories were introduced to generalize Hopf algebroids and groupoids. A quantum groupoid is a Hopf algebra with several objects.
2004	Stephan Stolz-Peter Teichner	Definition of nD QFT of degree p parametrized by a manifold.
2004	Stephan Stolz-Peter Teichner	Graeme Segal proposed in the 1980s to provide a geometric construction of elliptic cohomology (the precursor to tmf) as some kind of moduli space of CFTs. Stephan Stolz and Peter Teichner continued and expanded these ideas in a program to construct TMF as a moduli space of supersymmetric Euclidean field theories. They conjectured a Stolz-Teichner picture (analogy) between classifying spaces of cohomology theories in the chromatic filtration (de Rham cohomology,K-theory,Morava K-theories) and moduli spaces of supersymmetric QFTs parametrized by a manifold (proved in 0D and 1D).
2005	Peter Selinger	Dagger categories and dagger functors. Dagger categories seem to be part of a larger framework involving n-categories with duals.
2005	Peter Ozsváth-Zoltán Szabó	Knot Floer homology
2006	P. Carrasco-A.R. Garzon-E.M. Vitale	Categorical crossed modules
2006	Aslak Buan–Robert Marsh–Markus Reineke–Idun Reiten–Gordana Todorov	Cluster categories: Cluster categories are a special case of triangulated Calabi–Yau categories of Calabi–Yau dimension 2 and a generalization of cluster algebras.
2006	Jacob Lurie	Monumental book: Higher topos theory: In its 940 pages Jacob Lurie generalize the common concepts of category theory to higher categories and defines n-toposes, ∞-toposes, sheaves of n-types, ∞-sites, ∞-Yoneda lemma and proves Lurie characterization theorem for higher dimensional toposes. Luries theory of higher toposes can be interpreted as giving a good theory of sheaves taking values in ∞-categories. Roughly an ∞-topos is an ∞-category which looks like the ∞-category of all homotopy types. In a topos mathematics can be done. In a higher topos not only mathematics can be done but also "n-geometry", which is higher homotopy theory. The topos hypothesis is that the (n+1)-category nCat is a Grothendieck (n+1)-topos. Higher topos theory can also be used in a purely algebro-geometric way to solve various moduli problems in this setting.
2006	Marni Dee Sheppeard	Quantum toposes
2007	Bernhard Keller-Thomas Hugh	d-cluster categories
2007	Dennis Gaitsgory-Jacob Lurie	Presents a derived version of the geometric Satake equivalence and formulates a geometric Langlands duality for quantum groups. The geometric Satake equivalence realized the category of representations of the Langlands dual group ^LG in terms of spherical perverse sheaves (or D-modules) on the affine Grassmannian $Gr_G = G((t))/G[[t]]$ of the original group G.
2008	Ieke Moerdijk-Clemens Berger	Extends and improved the definition of Reedy category to become invariant under equivalence of categories.
2008	Michael J. Hopkins–Jacob Lurie	Sketch of proof of Baez-Dolan tangle hypothesis and Baez-Dolan cobordism hypothesis which classify extended TQFT in all dimensions.

Notes

[1] (http://www.PaulTaylor.EU/ASD/)

References

- nLab (http://ncatlab.org/nlab/list), just as a higher dimensional wikipedia, started in late 2008; see nLab
- Zhaohua Luo; Categorical geometry homepage (http://www.geometry.net/cg/index.html)
- John Baez, Aaron Lauda; A prehistory of n-categorical physics (http://math.ucr.edu/home/baez/history.pdf)
- Ross Street; An Australian conspectus of higher categories (http://www.maths.mq.edu.au/~street/Minneapolis.pdf)
- Elaine Landry, Jean-Pierre Marquis; Categories in context: historical, foundational, and philosophical (http://philmat.oxfordjournals.org/cgi/reprint/13/1/1)
- Jim Stasheff; A survey of cohomological physics (http://www.math.unc.edu/Faculty/jds/survey.pdf)
- John Bell; The development of categorical logic (http://publish.uwo.ca/~jbell/catlogprime.pdf)
- Jean Dieudonne; The historical development of algebraic geometry (http://www.joma.org/images/upload_library/22/Ford/Dieudonne.pdf)
- Charles Weibel; History of homological algebra (http://www.math.uiuc.edu/K-theory/0245/survey.pdf)
- Peter Johnstone; The point of pointless topology (http://projecteuclid.org/DPubS?verb=Display&version=1.0&service=UI&handle=euclid.bams/1183550014&page=record)
- Jim Stasheff; The pre-history of operads (http://citeseerx.ist.psu.edu/viewdoc/download;jsessionid=41C77863CA9DA3FE21D8C0FEE3E93BEE?doi=10.1.1.25.5089&rep=rep1&type=pdf)
- George Whitehead; Fifty years of homotopy theory (http://projecteuclid.org/DPubS/Repository/1.0/Disseminate?view=body&id=pdf_1&handle=euclid.bams/1183550012)
- Haynes Miller; The origin of sheaf theory (http://www-math.mit.edu/~hrm/papers/ss.ps)

List of important publications in mathematics

This is a list of **important publications** in mathematics, organized by field.

Some reasons why a particular publication might be regarded as important:

- **Topic creator** – A publication that created a new topic
- **Breakthrough** – A publication that changed scientific knowledge significantly
- **Introduction** – A publication that is a good introduction or survey of a topic
- **Influence** – A publication which has significantly influenced the world
- **Latest and greatest** – The current most advanced result in a topic

Algebra

Theory of equations

Al-Kitāb al-mukhtaṣar fī hīsāb al-ğabr wa'l-muqābala

- Muhammad ibn Mūsā al-Khwārizmī (820)

Description: The first book on the systematic algebraic solutions of linear and quadratic equations by the Persian scholar Muhammad ibn Mūsā al-Khwārizmī. The book is considered to be the foundation of modern algebra and Islamic mathematics. The word "algebra" itself is derived from the *al-Jabr* in the title of the book.

Ars Magna

- Gerolamo Cardano (1545)

Description: Provided the first published methods for solving cubic and quartic equations (due to Scipione del Ferro, Niccolò Fontana Tartaglia, and Lodovico Ferrari), and exhibited the first published calculations involving non-real complex numbers.[1]

Vollständige Anleitung zur Algebra

- Leonhard Euler (1770)

Description: Also known as Elements of Algebra, Euler's textbook on elementary algebra is one of the first to set out algebra in the modern form we would recognize today. The first volume deals with determinate equations, while the second part deals with Diophantine equations. The last section contains a proof of Fermat's Last Theorem for the case $n = 3$, making some valid assumptions regarding $\mathbb{Q}(\sqrt{-3})$ that Euler did not prove.[2]

Demonstratio nova theorematis omnem functionem algebraicam rationalem integram unius variabilis in factores reales primi vel secundi gradus resolvi posse

- Carl Friedrich Gauss (1799)

Description: Gauss' doctoral dissertation,[3] which contained a widely accepted (at the time) but incomplete proof[4] of the fundamental theorem of algebra.

Abstract algebra

Group theory

Réflexions sur la résolution algébrique des équations

- Joseph Louis Lagrange (1770)

Description: Made the prescient observation that the roots of the Lagrange resolvent of a polynomial equation are tied to permutations of the roots of the original equation, laying a more general foundation for what had previously been an ad hoc analysis and helping motivate the later development of the theory of permutation groups, group theory, and Galois theory. The Lagrange resolvent also introduced the discrete Fourier transform of order 3.

Articles Publiés par Galois dans les Annales de Mathématiques

- Journal de Mathematiques pures et Appliquées, II (1846)

Description: Posthumous publication of the mathematical manuscripts of Évariste Galois by Joseph Liouville. Included are Galois' papers *Mémoire sur les conditions de résolubilité des équations par radicaux* and *Des équations primitives qui sont solubles par radicaux*.

Traité des substitutions et des équations algébriques

- Camille Jordan (1870)

Online version: Online version [5]

Description: The first book on group theory, giving a then-comprehensive study of permutation groups and Galois theory. In this book, Jordan introduced the notion of a simple group and epimorphism (which he called *l'isomorphisme mériédrique*),[6] proved part of the Jordan–Hölder theorem, and discussed matrix groups over finite fields as well as the Jordan normal form.[7]

Theorie der Transformationsgruppen

- Sophus Lie, Friedrich Engel (1888–1893).

Publication data: 3 volumes, B.G. Teubner, Verlagsgesellschaft, mbH, Leipzig, 1888–1893. Volume 1 [8], Volume 2 [9], Volume 3 [9].

Description: The first comprehensive work on transformation groups, serving as the foundation for the modern theory of Lie groups.

Solvability of groups of odd order

- Walter Feit and John Thompson (1960)

Description: Gave a complete proof of the solvability of finite groups of odd order, establishing the long-standing Burnside conjecture that all finite non-abelian simple groups are of even order. Many of the original techniques used in this paper were used in the eventual classification of finite simple groups.

Homological algebra

Homological Algebra

- Henri Cartan and Samuel Eilenberg (1956)

Description: Provided the first fully-worked out treatment of abstract homological algebra, unifying previously disparate presentations of homology and cohomology for associative algebras, Lie algebras, and groups into a single theory.

Sur Quelques Points d'Algèbre Homologique

- Alexander Grothendieck (1957)

Description: Revolutionized homological algebra by introducing abelian categories and providing a general framework for Cartan and Eilenberg’s notion of derived functors.

Algebraic geometry

Theorie der Abelschen Functionen

- Bernhard Riemann (1857)

Publication data: *Journal für die Reine und Angewandte Mathematik*

Description: Developed the concept of Riemann surfaces and their topological properties beyond Riemann's 1851 thesis work, proved an index theorem for the genus (the original formulation of the Riemann-Hurwitz formula), proved the Riemann inequality for the dimension of the space of meromorphic functions with prescribed poles (the original formulation of the Riemann-Roch theorem), discussed birational transformations of a given curve and the dimension of the corresponding moduli space of inequivalent curves of a given genus, and solved more general inversion problems than those investigated by Abel and Jacobi. André Weil once wrote that this paper "*is one of the greatest pieces of mathematics that has ever been written; there is not a single word in it that is not of consequence.*"[10]

Faisceaux Algébriques Cohérents

- Jean-Pierre Serre

Publication data: *Annals of Mathematics*, 1955

Description: *FAC*, as it is usually called, was foundational for the use of sheaves in algebraic geometry, extending beyond the case of complex manifolds. Serre introduced Čech cohomology of sheaves in this paper, and, despite some technical deficiencies, revolutionized formulations of algebraic geometry. For example, the long exact sequence in sheaf cohomology allows one to show that some surjective maps of sheaves induce surjective maps on sections; specifically, these are the maps whose kernel (as a sheaf) has a vanishing first cohomology group. The dimension of a vector space of sections of a coherent sheaf is finite, in projective geometry, and such dimensions include many discrete invariants of varieties, for example Hodge numbers. While Grothendieck's derived functor cohomology has replaced Čech cohomology for technical reasons, actual calculations, such as of the cohomology of projective space, are usually carried out by Čech techniques, and for this reason Serre's paper remains important.

Géométrie Algébrique et Géométrie Analytique

- Jean-Pierre Serre (1956)

Description: In mathematics, algebraic geometry and analytic geometry are closely related subjects, where *analytic geometry* is the theory of complex manifolds and the more general analytic spaces defined locally by the vanishing of analytic functions of several complex variables. A (mathematical) theory of the relationship between the two was put in place during the early part of the 1950s, as part of the business of laying the foundations of algebraic geometry to include, for example, techniques from Hodge theory. (*NB* While analytic geometry as use of Cartesian coordinates is also in a sense included in the scope of algebraic geometry, that is not the topic being discussed in this article.) The major paper consolidating the theory was *Géometrie Algébrique et Géométrie Analytique* by Serre, now usually referred to as *GAGA*. A *GAGA-style result* would now mean any theorem of comparison, allowing passage between a category of objects from algebraic geometry, and their morphisms, and a well-defined subcategory of analytic geometry objects and holomorphic mappings.

Le théorème de Riemann-Roch, d'après A. Grothendieck

- Armand Borel, Jean-Pierre Serre (1958)

Description: Borel and Serre's exposition of Grothendieck's version of the Riemann Roch theorem, published after Grothendieck made it clear that he was not interested in writing up his own result. Grothendieck reinterpreted both sides of the formula that Hirzebruch proved in 1953 in the framework of morphisms between varieties, resulting in a sweeping generalization.[11] In his proof, Grothendieck broke new ground with his concept of Grothendieck groups, which led to the development of K-theory.[12]

Éléments de géométrie algébrique

- Alexander Grothendieck (1960–1967)

Description: Written with the assistance of Jean Dieudonné, this is Grothendieck's exposition of his reworking of the foundations of algebraic geometry. It has become the most important foundational work in modern algebraic geometry. The approach expounded in EGA, as these books are known, transformed the field and led to monumental advances.

Séminaire de géométrie algébrique

- Alexander Grothendieck et al.

Description: These seminar notes on Grothendieck's reworking of the foundations of algebraic geometry report on work done at IHÉS starting in the 1960s. SGA 1 dates from the seminars of 1960–1961, and the last in the series, SGA 7, dates from 1967–1969. In contrast to EGA, which is intended to set foundations, SGA describes ongoing research as it unfolded in Grothendieck’s seminar; as a result, it is quite difficult to read, since many of the more elementary and foundational results were relegated to EGA. One of the major results building on the results in SGA is Pierre Deligne's proof of the last of the open Weil conjectures in the early 1970s. Other authors who worked on one or several volumes of SGA include Michel Raynaud, Michael Artin, Jean-Pierre Serre, Jean-Louis Verdier, Pierre Deligne, and Nicholas Katz.

Number theory

De fractionibus continuis dissertatio

- Leonhard Euler (1744)

Description: First presented in 1737, this paper [13] provided the first then-comprehensive account of the properties of continued fractions. It also contains the first proof that the number e is irrational.[14]

Recherches d'Arithmétique

- Joseph Louis Lagrange (1775)

Description: Developed a general theory of binary quadratic forms to handle the general problem of when an integer is representable by the form $ax^2 + by^2 + cxy$. This included a reduction theory for binary quadratic forms, where he proved that every form is equivalent to a certain canonically chosen reduced form.[15] [16]

Disquisitiones Arithmeticae

- Carl Friedrich Gauss (1801)

Description: The *Disquisitiones Arithmeticae* is a profound and masterful book on number theory written by German mathematician Carl Friedrich Gauss and first published in 1801 when Gauss was 24. In this book Gauss brings together results in number theory obtained by mathematicians such as Fermat, Euler, Lagrange and Legendre and adds many important new results of his own. Among his contributions was the first complete proof known of the Fundamental theorem of arithmetic, the first two published proofs of the law of quadratic reciprocity, a deep investigation of binary quadratic forms going beyond Lagrange's work in Recherches d'Arithmétique, a first appearance of Gauss sums, cyclotomy, and the theory of constructible polygons with a particular application to the constructibility of the regular 17-gon. Of note, in section V, article 303 of Disquisitiones, Gauss summarized his calculations of class numbers of imaginary quadratic number fields, and in fact found all imaginary quadratic number fields of class numbers 1, 2, and 3 (confirmed in 1986) as he had conjectured.[17] In section VII, article 358, Gauss proved what can be interpreted as the first non-trivial case of the Riemann Hypothesis for curves over finite fields (the Hasse-Weil theorem).[18]

Beweis des Satzes, daß jede unbegrenzte arithmetische Progression, deren erstes Glied und Differenz ganze Zahlen ohne gemeinschaftlichen Factor sind, unendlich viele Primzahlen enthä lt

- Johann Peter Gustav Lejeune Dirichlet (1837)

Description: Pioneering paper in analytic number theory, which introduced Dirichlet characters and their L-functions to establish Dirichlet's theorem on arithmetic progressions.[19] In subsequent publications, Dirichlet used these tools to determine, among other things, the class number for quadratic forms.

Über die Anzahl der Primzahlen unter einer gegebenen Grösse

- Bernhard Riemann (1859)

Description: *Über die Anzahl der Primzahlen unter einer gegebenen Grösse* (or *On the Number of Primes Less Than a Given Magnitude*) is a seminal 8-page paper by Bernhard Riemann published in the November 1859 edition of the *Monthly Reports of the Berlin Academy*. Although it is the only paper he ever published on number theory, it contains ideas which influenced dozens of researchers during the late 19th century and up to the present day. The paper consists primarily of definitions, heuristic arguments, sketches of proofs, and the application of powerful analytic methods; all of these have become essential concepts and tools of modern analytic number theory. It also contains the famous Riemann Hypothesis, one of the most important open problems in mathematics.

Vorlesungen über Zahlentheorie

- P.G.L. Dirichlet and Richard Dedekind

Description: *Vorlesungen über Zahlentheorie* (*Lectures on Number Theory*) is a textbook of number theory written by German mathematicians P.G.L. Dirichlet and Richard Dedekind, and published in 1863. The *Vorlesungen* can be seen as a watershed between the classical number theory of Fermat, Jacobi and Gauss, and the modern number theory of Dedekind, Riemann and Hilbert. Dirichlet does not explicitly recognise the concept of the group that is central to modern algebra, but many of his proofs show an implicit understanding of group theory

Zahlbericht

- David Hilbert (1897)

Description: Unified and made accessible many of the developments in algebraic number theory made during the nineteenth century. Although criticized by André Weil (who stated "*more than half of his famous Zahlbericht is little more than an account of Kummer’s number-theoretical work, with inessential improvements*")[20] and Emmy Noether,[21] it was highly influential for many years following its publication.

Fourier Analysis in Number Fields and Hecke's Zeta-Functions

- John Tate (1950)

Description: Generally referred to simply as *Tate's Thesis*, Tate's Princeton Ph.D. thesis, under Emil Artin, is a reworking of Erich Hecke's theory of zeta- and *L*-functions in terms of Fourier analysis on the adeles. The introduction of these methods into number theory made it possible to formulate extensions of Hecke's results to more general *L*-functions such as those arising from automorphic forms.

Automorphic Forms on GL(2)

- Hervé Jacquet and Robert Langlands (1970)

Description: This publication offers evidence towards Langlands' conjectures by reworking and expanding the classical theory of modular forms and their *L*-functions through the introduction of representation theory.

La conjecture de Weil. I.

- Pierre Deligne (1974)

Description: Proved the Riemann hypothesis for varieties over finite fields, settling the last of the open Weil conjectures.

Endlichkeitssätze für abelsche Varietäten über Zahlkörpern

- Gerd Faltings (1983)

Description: Faltings proves a collection of important results in this paper, the most famous of which is the first proof of the Mordell conjecture (a conjecture dating back to 1922). Other theorems proved in this paper include an instance of the Tate conjecture (relating the homomorphisms between two abelian varieties over a number field to the homomorphisms between their Tate modules) and some finiteness results concerning abelian varieties over number fields with certain properties.

Modular Elliptic Curves and Fermat's Last Theorem

- Andrew Wiles (1995)

Description: This article proceeds to prove a special case of the Shimura-Taniyama conjecture through the study of the deformation theory of Galois representations. This in turn implies the famed Fermat's Last Theorem. The proof's method of identification of a deformation ring with a Hecke algebra (now referred to as an $R=T$ theorem) to prove modularity lifting theorems has been an influential development in algebraic number theory.

The geometry and cohomology of some simple Shimura varieties

- Michael Harris and Richard Taylor (2001)

Description: Harris and Taylor provide the first proof of the local Langlands conjecture for GL(n). As part of the proof, this monograph also makes an in depth study of the geometry and cohomology of certain Shimura varieties at primes of bad reduction.

Analysis

Introductio in analysin infinitorum

- Leonhard Euler (1748)

Description: The eminent historian of mathematics Carl Boyer once called Euler's *Introductio in analysin infinitorum* the greatest modern textbook in mathematics.[22] Published in two volumes,[23] [24] this book more than any other work succeeded in establishing analysis as a major branch of mathematics, with a focus and approach distinct from that used in geometry and algebra.[25] Notably, Euler identified functions rather than curves to be the central focus in his book.[26] Logarithmic, exponential, trigonometric, and transcendental functions were covered, as were expansions into partial fractions, evaluations of ζ(2k) for k a positive integer between 1 and 13, infinite series-infinite product formulas,[22] continued fractions, and partitions of integers.[27] In this work, Euler proved that every rational number can be written as a finite continued fraction, that the continued fraction of an irrational number is infinite, and derived continued fraction expansions for e and $\sqrt{e}$.[23] This work also contains a statement of Euler's formula and a statement of the pentagonal number theorem, which he had discovered earlier and would publish a proof for in 1751.

Calculus

Yuktibhāṣā

- Jyeshtadeva (1501)

Description: Written in India in 1501, this was the world's first calculus text. "This work laid the foundation for a complete system of fluxions"[28] and served as a summary of the Kerala School's achievements in calculus, trigonometry and mathematical analysis, most of which were earlier discovered by the 14th century mathematician Madhava. It's possible that this text influenced the later development of calculus in Europe. Some of its important developments in calculus include: the fundamental ideas of differentiation and integration, the derivative, differential equations, term by term integration, numerical integration by means of infinite series, the relationship between the area of a curve and its integral, and the mean value theorem.

Nova methodus pro maximis et minimis, itemque tangentibus, quae nec fractas nec irrationales quantitates moratur, et singulare pro illi calculi genus

- Gottfried Leibniz (1684)

Description: Leibniz's first publication on differential calculus, containing the now familiar notation for differentials as well as rules for computing the derivatives of powers, products and quotients.

Philosophiae Naturalis Principia Mathematica

- Isaac Newton

Description: The ***Philosophiae Naturalis Principia Mathematica*** (Latin: "mathematical principles of natural philosophy", often *Principia* or *Principia Mathematica* for short) is a three-volume work by Isaac Newton published on 5 July 1687. Perhaps the most influential scientific book ever published, it contains the statement of Newton's laws of motion forming the foundation of classical mechanics as well as his law of universal gravitation, and derives Kepler's laws for the motion of the planets (which were first obtained empirically). Here was born the practice, now so standard we identify it with science, of explaining nature by postulating mathematical axioms and demonstrating that their conclusion are observable phenomena. In formulating his physical theories, Newton freely used his unpublished work on calculus. When he submitted Principia for publication, however, Newton chose to recast the majority of his proofs as geometric arguments.[29]

Institutiones calculi differentialis cum eius usu in analysi finitorum ac doctrina serierum

- Leonhard Euler (1755)

Description: Published in two books,[30] Euler's textbook on differential calculus presented the subject in terms of the function concept, which he had introduced in his 1748 *Introductio in analysin infinitorum*. This work opens with a study of the calculus of finite differences and makes a thorough investigation of how differentiation behaves under substitutions.[1] Also included is a systematic study of Bernoulli polynomials and the Bernoulli numbers (naming them as such), a demonstration of how the Bernoulli numbers are related to the coefficients in the Euler–Maclaurin formula and the values of $\zeta(2n)$,[31] a further study of Euler's constant (including its connection to the gamma function), and an application of partial fractions to differentiation.[32]

Über die Darstellbarkeit einer Function durch eine trigonometrische Reihe

- Bernhard Riemann (1867)

Description: Written in 1853, Riemann's work on trigonometric series was published posthumously. In it, he extended Cauchy’s definition of the integral to that of the Riemann integral, allowing some functions with dense subsets of discontinuities on an interval to be integrated (which he demonstrated by an example).[33] He also stated the Riemann series theorem,[33] proved the Riemann-Lebesgue lemma for the case of bounded Riemann integrable functions,[34] and developed the Riemann localization principle.[35]

Intégrale, longueur, aire

- Henri Lebesgue (1901)

Description: Lebesgue's doctoral dissertation, summarizing and extending his research to date regarding his development of measure theory and the Lebesgue integral.

Complex analysis

Grundlagen für eine allgemeine Theorie der Functionen einer veränderlichen complexen Grösse

- Bernhard Riemann (1851)

Description: Riemann's doctoral dissertation introduced the notion of a Riemann surface, conformal mapping, simple connectivity, the Riemann sphere, the Laurent series expansion for functions having poles and branch points, and the Riemann mapping theorem.

Functional analysis

Théorie des opérations linéaires

- Stefan Banach (1932; originally published 1931 in Polish under the title *Teorja operacyj*.)

Description: The first mathematical monograph on the subject of linear metric spaces, bringing the abstract study of functional analysis to the wider mathematical community. The book introduced the ideas of a normed space and the notion of a so-called *B*-space, a complete normed space. The *B*-spaces are now called Banach spaces and are one of the basic objects of study in all areas of modern mathematical analysis. Banach also gave proofs of versions of the open mapping theorem, closed graph theorem, and Hahn–Banach theorem.

Fourier analysis

Mémoire sur la propagation de la chaleur dans les corps solides

- Joseph Fourier (1807)[36]

Description: Introduced Fourier analysis, specifically Fourier series. Key contribution was to not simply use trigonometric series, but to model *all* functions by trigonometric series.

> “$\varphi(y) = a\cos\frac{\pi y}{2} + a'\cos 3\frac{\pi y}{2} + a''\cos 5\frac{\pi y}{2} + \cdots.$
>
> Multiplying both sides by $\cos(2i+1)\frac{\pi y}{2}$, and then integrating from $y=-1$ to $y=+1$ yields:
>
> $a_i = \int_{-1}^{1} \varphi(y)\cos(2i+1)\frac{\pi y}{2}\,dy.$”

When Fourier submitted his paper in 1807, the committee (which included Lagrange, Laplace, Malus and Legendre, among others) concluded: ...*the manner in which the author arrives at these equations is not exempt of difficulties and [...] his analysis to integrate them still leaves something to be desired on the score of generality and even rigour.* Making Fourier series rigorous, which in detail took over a century, led directly to a number of developments in analysis, notably the rigorous statement of the integral via the Dirichlet integral and later the Lebesgue integral.

Sur la convergence des séries trigonométriques qui servent à représenter une fonction arbitraire entre des limites données

- Johann Peter Gustav Lejeune Dirichlet (1829, expanded German edition in 1837)

Description: In his habilitation thesis on Fourier series, Riemann characterized this work of Dirichlet as "*the first profound paper about the subject*".[37] This paper gave the first rigorous proof of the convergence of Fourier series under fairly general conditions (piecewise continuity and monotonicity) by considering partial sums, which Dirichlet transformed into a particular Dirichlet integral involving what is now called the Dirichlet kernel. This paper introduced the nowhere continuous Dirichlet function and an early version of the Riemann-Lebesgue lemma.[38]

On convergence and growth of partial sums of Fourier series

- Lennart Carleson (1966)

Description: Settled Lusin's conjecture that the Fourier expansion of any L^2 function converges almost everywhere.

Geometry

Baudhayana Sulba Sutra

- Baudhayana

Description: Written around the 8th century BC, this is one of the oldest geometrical texts. It laid the foundations of Indian mathematics and was influential in South Asia and its surrounding regions, and perhaps even Greece. Among the important geometrical discoveries included in this text are: the earliest list of Pythagorean triples discovered algebraically, the earliest statement of the Pythagorean theorem, geometric solutions of linear equations, several approximations of π, the first use of irrational numbers, and an accurate computation of the square root of 2, correct to a remarkable five decimal places. Though this was primarily a geometrical text, it also contained some important algebraic developments, including the earliest use of quadratic equations of the forms $ax^2 = c$ and $ax^2 + bx = c$, and integral solutions of simultaneous Diophantine equations with up to four unknowns.

Euclid's Elements

- Euclid

Publication data: c. 300 BC

Online version: Interactive Java version [39]

Description: This is often regarded as not only the most important work in geometry but one of the most important works in mathematics. It contains many important results in geometry, number theory and the first algorithm as well. More than any specific result in the publication, it seems that the major achievement of this publication is the popularization of logic and mathematical proof as a method of solving problems.

The Nine Chapters on the Mathematical Art

- Unknown author

Description: This was a Chinese mathematics book, mostly geometric, composed during the Han Dynasty, perhaps as early as 200 BC. It remained the most important textbook in China and East Asia for over a thousand years, similar to the position of Euclid's *Elements* in Europe. Among its contents: Linear problems solved using the principle known later in the West as the *rule of false position*. Problems with several unknowns, solved by a principle similar to Gaussian elimination. Problems involving the principle known in the West as the Pythagorean theorem. The earliest solution of a matrix using a method equivalent to the modern method.

The Conics

- Apollonius of Perga

Description: The Conics was written by Apollonius of Perga, a Greek mathematician. His innovative methodology and terminology, especially in the field of conics, influenced many later scholars including Ptolemy, Francesco Maurolico, Isaac Newton, and René Descartes. It was Apollonius who gave the ellipse, the parabola, and the hyperbola the names by which we know them.

La Géométrie

- René Descartes

Description: La Géométrie was published in 1637 and written by René Descartes. The book was influential in developing the Cartesian coordinate system and specifically discussed the representation of points of a plane, via real numbers; and the representation of curves, via equations.

Grundlagen der Geometrie

- David Hilbert

Publication data: Hilbert, David (1899). *Grundlagen der Geometrie*. Teubner-Verlag Leipzig. ISBN 140202777X.

Description: Hilbert's axiomatization of geometry, whose primary influence was in its pioneering approach to metamathematical questions including the use of models to prove axiom independence and the importance of establishing the consistency and completeness of an axiomatic system.

Regular Polytopes

- H.S.M. Coxeter

Description: *Regular Polytopes* is a comprehensive survey of the geometry of regular polytopes, the generalisation of regular polygons and regular polyhedra to higher dimensions. Originating with an essay entitled *Dimensional Analogy* written in 1923, the first edition of the book took Coxeter 24 years to complete. Originally written in 1947, the book was updated and republished in 1963 and 1973.

Differential geometry

Recherches sur la courbure des surfaces

- Leonard Euler (1760)

Publication data: Mémoires de l'académie des sciences de Berlin **16** (1760) pp. 119–143; published 1767. (Full text [40] and an English translation available from the Dartmouth Euler archive.)

Description: Established the theory of surfaces, and introduced the idea of principal curvatures, laying the foundation for subsequent developments in the differential geometry of surfaces.

Disquisitiones generales circa superficies curvas

- Carl Friedrich Gauss (1827)

Publication data: "Disquisitiones generales circa superficies curvas" [41], *Commentationes Societatis Regiae Scientiarum Gottingesis Recentiores* Vol. **VI** (1827), pp. 99–146; "General Investigations of Curved Surfaces [42]" (published 1965) Raven Press, New York, translated by A.M.Hiltebeitel and J.C.Morehead.

Description: Groundbreaking work in differential geometry, introducing the notion of Gaussian curvature and Gauss' celebrated Theorema Egregium.

Über die Hypothesen, welche der Geometrie zu Grunde Liegen

- Bernhard Riemann (1854)

Publication data: "Über die Hypothesen, welche der Geometrie zu Grunde Liegen" [43], *Abhandlungen der Königlichen Gesellschaft der Wissenschaften zu Göttingen*, Vol. 13, 1867.

Description: Riemann's famous Habiltationsvortrag, in which he introduced the notions of a manifold, Riemannian metric, and curvature tensor.

Leçons sur la théorie génerale des surfaces

- Gaston Darboux

Publication data: Darboux, Gaston (1887,1889,1896). *Leçons sur la théorie génerale des surfaces: Volume I* [44], *Volume II* [45], *Volume III* [46], *Volume IV* [46]. Gauthier-Villars.

Description: A treatise covering virtually every aspect of the 19th century differential geometry of surfaces.

Topology

Analysis situs

- Henri Poincaré (1895, 1899–1905)

Description: Poincaré's Analysis situs and his Compléments à l'Analysis Situs laid the general foundations for algebraic topology. In these papers, Poincaré introduced the notions of homology and the fundamental group, provided an early formulation of Poincaré duality, gave the Euler-Poincaré characteristic for chain complexes, and mentioned several important conjectures including the Poincaré conjecture.

Grundzüge der Mengenlehre

- Felix Hausdorff (1914)
- Reprinted and commented in: *Gesammelte Werke Bd. 2* / Herausgegeben von E. Brieskorn, S.D.Chatterji *et al.*. – Berlin, 2002

Description: This book founded (general) topology by giving the axioms for a (Hausdorff) topological space.

L'anneau d'homologie d'une représentation, *Structure de l'anneau d'homologie d'une représentation*

- Jean Leray (1946)

Description: These two Comptes Rendus notes of Leray from 1946 introduced the novel concepts of sheafs, sheaf cohomology, and spectral sequences, which he had developed during his years of captivity as a prisoner of war. Leray's announcements and applications (published in other Comptes Rendus notes from 1946) drew immediate attention from other mathematicians. Subsequent clarification, development, and generalization by Henri Cartan, Jean-Louis Koszul, Armand Borel, Jean-Pierre Serre, and Leray himself allowed these concepts to be understood and applied to many other areas of mathematics.[47] Dieudonné would later write that these notions created by Leray "*undoubtedly rank at the same level in the history of mathematics as the methods invented by Poincaré and Brouwer*".[12]

Quelques propriétés globales des variétés differentiables

- René Thom (1954)

Description: In this paper, Thom proved the Thom transversality theorem, introduced the notions of oriented and unoriented cobordism, and demonstrated that cobordism groups could be computed as the homotopy groups of certain Thom spaces. Thom completely characterized the unoriented cobordism ring and achieved strong results for several problems, including Steenrod's problem on the realization of cycles.[12] [48]

Category theory

General theory of natural equivalences

- Samuel Eilenberg and Saunders Mac Lane (1945)

Description: The first paper on category theory. Mac Lane later wrote in *Categories for the Working Mathematician* that he and Eilenberg introduced categories so that they could introduce functors, and they introduced functors so that they could introduce natural equivalences. Prior to this paper, "natural" was used in an informal and imprecise way to designate constructions that could be made without making any choices. Afterwards, "natural" had a precise meaning which occurred in a wide variety of contexts and had powerful and important consequences.

Categories for the Working Mathematician

- Saunders Mac Lane (1971, second edition 1998)

Description: Saunders Mac Lane, one of the founders of category theory, wrote this exposition to bring categories to the masses. Mac Lane brings to the fore the important concepts that make category theory useful, such as adjoint functors and universal properties.

Set theory

Über eine Eigenschaft des Inbegriffes aller reellen algebraischen Zahlen

- Georg Cantor (1874)

Description: Contains the first proof that the set of all real numbers is uncountable; also contains a proof that the set of algebraic numbers is denumerable.

Grundzüge der Mengenlehre

- Felix Hausdorff

Description: First published in 1914, this was the first comprehensive introduction to set theory. Besides the systematic treatment of known results in set theory, the book also contains chapters on measure theory and topology, which were then still considered parts of set theory. Here Hausdorff presents and develops highly original material which was later to become the basis for those areas.

The consistency of the axiom of choice and of the generalized continuum-hypothesis with the axioms of set theory

- Kurt Gödel (1938)

Description: Gödel proves the results of the title. Also, in the process, introduces the class L of constructible sets, a major influence in the development of axiomatic set theory.

The Independence of the Continuum Hypothesis

- Paul J. Cohen (1963, 1964)

Description: Cohen's breakthrough work proved the independence of the continuum hypothesis and axiom of choice with respect to Zermelo-Fraenkel set theory. In proving this Cohen introduced the concept of *forcing* which led to many other major results in axiomatic set theory.

Logic

Begriffsschrift

- Gottlob Frege (1879)

Description: Published in 1879, the title ***Begriffsschrift*** is usually translated as *concept writing* or *concept notation*; the full title of the book identifies it as "*a formula language, modelled on that of arithmetic, of pure thought*". Frege's motivation for developing his formal logical system was similar to Leibniz's desire for a *calculus ratiocinator*. Frege defines a logical calculus to support his research in the foundations of mathematics. ***Begriffsschrift*** is both the name of the book and the calculus defined therein. It was arguably the most significant publication in logic since Aristotle.

Formulario mathematico

- Giuseppe Peano (1895)

Description: First published in 1895, the **Formulario mathematico** was the first mathematical book written entirely in a formalized language. It contained a description of mathematical logic and many important theorems in other branches of mathematics. Many of the notations introduced in the book are now in common use.

Principia Mathematica

- Bertrand Russell and Alfred North Whitehead (1910–1913)

Description: The ***Principia Mathematica*** is a three-volume work on the foundations of mathematics, written by Bertrand Russell and Alfred North Whitehead and published in 1910–1913. It is an attempt to derive all mathematical truths from a well-defined set of axioms and inference rules in symbolic logic. The questions remained whether a contradiction could be derived from the Principia's axioms, and whether there exists a mathematical statement which could neither be proven nor disproven in the system. These questions were settled, in a rather surprising way, by Gödel's incompleteness theorem in 1931.

Über formal unentscheidbare Sätze der Principia Mathematica und verwandter Systeme, I

(On Formally Undecidable Propositions of Principia Mathematica and Related Systems)

- Kurt Gödel (1931)

Online version: Online version [49]

Description: In mathematical logic, **Gödel's incompleteness theorems** are two celebrated theorems proved by Kurt Gödel in 1931. The first incompleteness theorem states:

> For any formal system such that (1) it is ω-consistent (omega-consistent), (2) it has a recursively definable set of axioms and rules of derivation, and (3) every recursive relation of natural numbers is definable in it, there exists a formula of the system such that, according to the intended interpretation of the system, it expresses a truth about natural numbers and yet it is not a theorem of the system.

Combinatorics

On sets of integers containing no k elements in arithmetic progression

- Endre Szemerédi (1975)

Description: Settled a conjecture of Paul Erdős and Paul Turán that if a sequence of natural numbers has positive upper density then it contains arbitrarily long arithmetic progressions. Szemerédi's solution has been described as a "masterpiece of combinatorics"[50] and it introduced new ideas and tools to the field including the Szemerédi regularity lemma.

Graph theory

Solutio problematis ad geometriam situs pertinentis

- Leonhard Euler (1741)
- Euler's original publication [51] (in Latin)

Description: Euler's solution of the Königsberg bridge problem in *Solutio problematis ad geometriam situs pertinentis* (*The solution of a problem relating to the geometry of position*) is considered to be the first theorem of graph theory.

On the evolution of random graphs

- Paul Erdös and Alfréd Rényi (1960)

Description: Provides a detailed discussion of sparse random graphs, including distribution of components, occurrence of small subgraphs, and phase transitions.[52]

Network Flows and General Matchings

- Ford, L., & Fulkerson, D.
- Flows in Networks. Prentice-Hall, 1962.

Description: Ford and Fulkerson paper on Network Flows. The algorithm along with many ideas on flow-based models can be found in their book.

Computational complexity theory

See List of important publications in theoretical computer science.

Probability theory

See list of important publications in statistics.

Game theory

Zur Theorie der Gesellschaftsspiele

- John von Neumann (1928)

Description: Went well beyond Émile Borel's initial investigations into strategic two-person game theory by proving the minimax theorem for two-person, zero-sum games.

Theory of Games and Economic Behavior

- Oskar Morgenstern, John von Neumann (1944)

Description: This book led to the investigation of modern game theory as a prominent branch of mathematics. This profound work contained the method for finding optimal solutions for two-person zero-sum games.

Equilibrium Points in N-person Games

- John Forbes Nash
- *Proceedings of the National Academy of Sciences* 36 (1950), 48–49. MR0031701
- "Equilibrium Points in N-person Games" [53]

Description: Nash equilibrium

On Numbers and Games

- John Horton Conway

Description: The book is in two, {0,1|}, parts. The zeroth part is about numbers, the first part about games – both the values of games and also some real games that can be played such as Nim, Hackenbush, Col and Snort amongst the many described.

Winning Ways for your Mathematical Plays

- Elwyn Berlekamp, John Conway and Richard K. Guy

Description: A compendium of information on mathematical games. It was first published in 1982 in two volumes, one focusing on Combinatorial game theory and surreal numbers, and the other concentrating on a number of specific games.

Fractals

How Long Is the Coast of Britain? Statistical Self-Similarity and Fractional Dimension

- Benoît Mandelbrot

Description: A discussion of self-similar curves that have fractional dimensions between 1 and 2. These curves are examples of fractals, although Mandelbrot does not use this term in the paper, as he did not coin it until 1975. Shows Mandelbrot's early thinking on fractals, and is an example of the linking of mathematical objects with natural forms that was a theme of much of his later work.

Numerical analysis

Numerical linear algebra

The Algebraic Eigenvalue Problem

- James H. Wilkinson (1965)

Description:

Optimization

Method of Fluxions

- Isaac Newton

Description: *Method of Fluxions* was a book written by Isaac Newton. The book was completed in 1671, and published in 1736. Within this book, Newton describes a method (the Newton-Raphson method) for finding the real zeroes of a function.

Essai d'une nouvelle méthode pour déterminer les maxima et les minima des formules intégrales indéfinies

- Joseph Louis Lagrange (1761)

Description: Major early work on the calculus of variations, building upon some of Lagrange's prior investigations as well as those of Euler. Contains investigations of minimal surface determination as well as the initial appearance of Lagrange multipliers.

Математические методы организации и планирования производства

- Leonid Kantorovich (1939) "[The Mathematical Method of Production Planning and Organization]" (in Russian).

Description: Kantorovich wrote the first paper on production planning, which used Linear Programs as the model. He proposed the simplex algorithm as a systematic procedure to solve these Linear Programs. He received the Nobel prize for this work in 1975.

Decomposition Principle for Linear Programs

- George Dantzig and P. Wolfe
- Operations Research 8:101–111, 1960.

Description: Dantzig's is considered the father of Linear Programming in the western world. He independently invented the simplex algorithm. Dantzig and Wolfe worked on decomposition algorithms for large scale linear programs in factory and production planning.

How good is the simplex algorithm?

- Victor Klee and George J. Minty
- In: O. Shisha (ed.) Inequalities III, Academic Press (1972) 159–175.

Description: Klee and Minty gave an example showing that the simplex method can take exponentially many steps to solve a linear program if it chooses the greedy ascent rule.

Полиномиальный алгоритм в линейном программировании

- Khachiyan, Leonid Genrikhovich (1979). "[A polynomial algorithm for linear programming]" (in Russian). *Doklady Akademii Nauk SSSR* **244**: 1093–1096..

Description: Khachiyan's work on Ellipsoid method. This was the first polynomial time algorithm for linear programming.

New polynomial-time algorithm for linear programming

- Karmarkar, N.
- Combinatorica 4, 373–395, 1984.

Description: Karmarkars path-breaking work on Interior-Point algorithms for Linear Programming.

Interior Point Polynomial Algorithms in Convex Programming

- Yurii Nesterov and A. Nemirovski
- Philadelphia : Society for Industrial and Applied Mathematics, 1994. (SIAM Studies in Applied Mathematics).

Description: Nesterov and Nemirovski's work on self-concordant barriers and interior-point methods for general convex programming. All their series of papers (both individual and combined) is compiled more coherently in the following "bible" of convex optimization.

Early manuscripts

These are publications that are not necessarily relevant to a mathematician nowadays, but are nonetheless important publications in the history of mathematics.

Rhind Mathematical Papyrus

- Ahmes (scribe)

Description: It is one of the oldest mathematical texts, dating to the Second Intermediate Period of ancient Egypt. It was copied by the scribe Ahmes (properly *Ahmose*) from an older Middle Kingdom papyrus. It laid the foundations of Egyptian mathematics and in turn, later influenced Greek and Hellenistic mathematics. Besides describing how to obtain an approximation of π only missing the mark by less than one per cent, it is describes one of the earliest attempts at squaring the circle and in the process provides persuasive evidence against the theory that the Egyptians deliberately built their pyramids to enshrine the value of π in the proportions. Even though it would be a strong overstatement to suggest that the papyrus represents even rudimentary attempts at analytical geometry, Ahmes did make use of a kind of an analogue of the cotangent.

Archimedes Palimpsest

- Archimedes of Syracuse

Description: Although the only mathematical tools at its author's disposal were what we might now consider secondary-school geometry, he used those methods with rare brilliance, explicitly using infinitesimals to solve problems that would now be treated by integral calculus. Among those problems were that of the center of gravity of a solid hemisphere, that of the center of gravity of a frustum of a circular paraboloid, and that of the area of a region bounded by a parabola and one of its secant lines. For explicit details of the method used, see Archimedes' use of infinitesimals.

The Sand Reckoner

- Archimedes of Syracuse

Online version: Online version [54]

Description: The first known (European) system of number-naming that can be expanded beyond the needs of everyday life.

Textbooks

Synopsis of Pure Mathematics

- G. S. Carr

Description: Contains over 6000 theorems of mathematics, assembled by George Shoobridge Carr for the purpose of training students in the art of mathematics, studied extensively by Ramanujan. (first half here) [55] It was one of the few books that attempts to summarize the entirety of known mathematics.

Arithmetick: or, The Grounde of Arts

- Robert Recorde

Description: Written in 1542, it was the first really popular arithmetic book written in the English Language.

Cocker's Arithmetick

- Edward Cocker (authorship disputed)

Description: Textbook of arithmetic published in 1678 by John Hawkins, who claimed to have edited manuscripts left by Edward Cocker, who had died in 1676. This influential mathematics textbook used to teach arithmetic in schools in the United Kingdom for over 150 years.

The Schoolmaster's Assistant, Being a Compendium of Arithmetic both Practical and Theoretical

- Thomas Dilworth

Description: An early and popular English arithmetic textbook published in America in the 18th century. The book reached from the introductory topics to the advanced in five sections.

Geometry

- Andrey Kiselyov

Publication data: 1892

Description: The most widely-used and influential textbook in Russian mathematics. (See Kiselyov page and MAA review [56].)

A Course of Pure Mathematics

- G. H. Hardy

Description: A classic textbook in introductory mathematical analysis, written by G. H. Hardy. It was first published in 1908, and went through many editions. It was intended to help reform mathematics teaching in the UK, and more specifically in the University of Cambridge, and in schools preparing pupils to study mathematics at Cambridge. As such, it was aimed directly at "scholarship level" students — the top 10% to 20% by ability. The book contains a large number of difficult problems. The content covers introductory calculus and the theory of infinite series.

Moderne Algebra

- B. L. van der Waerden

Description: The first introductory textbook (graduate level) expounding the abstract approach to algebra developed by Emil Artin and Emmy Noether. First published in German in 1931 by Springer Verlag. A later English translation was published in 1949 by Frederick Ungar Publishing Company.

Algebra

- Saunders Mac Lane and Garrett Birkhoff

Description: A definitive introductory text for abstract algebra using a category theoretic approach. Both a rigorous introduction from first principles, and a reasonably comprehensive survey of the field.

Algebraic Geometry

- Robin Hartshorne

Description: The first comprehensive introductory (graduate level) text in algebraic geometry that used the language of schemes and cohomology. Published in 1977, it lacks aspects of the scheme language which are nowadays considered central, like the functor of points.

Naive Set Theory

- Paul Halmos

Description: An undergraduate introduction to not-very-naive set theory which has lasted for decades. It is still considered by many to be the best introduction to set theory for beginners. While the title states that it is naive, which is usually taken to mean without axioms, the book does introduce all the axioms of Zermelo-Fraenkel set theory and gives correct and rigorous definitions for basic objects. Where it differs from a "true" axiomatic set theory book is its character: There are no long-winded discussions of axiomatic minutiae, and there is next to nothing about topics like large cardinals. Instead it aims, and succeeds, in being intelligible to someone who has never thought about set theory before.

Cardinal and Ordinal Numbers

- Waclaw Sierpinski

Description: The *nec plus ultra* reference for basic facts about cardinal and ordinal numbers. If you have a question about the cardinality of sets occurring in everyday mathematics, the first place to look is this book, first published in the early 1950s but based on the author's lectures on the subject over the preceding 40 years.

Set Theory: An Introduction to Independence Proofs

- Kenneth Kunen

Description: This book is not really for beginners, but graduate students with some minimal experience in set theory and formal logic will find it a valuable self-teaching tool, particularly in regard to forcing. It is far easier to read than a true reference work such as Jech, *Set Theory*. It may be the best textbook from which to learn forcing, though it has the disadvantage that the exposition of forcing relies somewhat on the earlier presentation of Martin's axiom.

Topologie

- Pavel Sergeevich Alexandrov
- Heinz Hopf

Description: First published round 1935, this text was a pioneering "reference" text book in topology, already incorporating many modern concepts from set-theoretic topology, homological algebra and homotopy theory.

General Topology

- John L. Kelley

Description: First published in the mid-1950s,for many years the only introductory graduate level textbook in the U.S.A. teaching the basics of point set, as opposed to algebraic, topology. Prior to this the material, essential for advanced study in many fields, was only available in bits and pieces from texts on other topics or journal articles.

Topology from the Differentiable Viewpoint

- John Milnor

Description: This short book introduces the main concepts of differential topology in Milnor's lucid and concise style. While the book does not cover very much, its topics are explained beautifully in a way that illuminates all their details.

Number Theory, An approach through history from Hammurapi to Legendre

- André Weil

Description: An historical study of number theory, written by one of the 20th century's greatest researchers in the field. The book covers some thirty six centuries of arithmetical work but the bulk of it is devoted to a detailed study and exposition of the work of Fermat, Euler, Lagrange, and Legendre. The author wishes to take the reader into the workshop of his subjects to share their successes and failures. A rare opportunity to see the historical development of a subject through the mind of one of its greatest practitioners.

An Introduction to the Theory of Numbers

- G. H. Hardy and E. M. Wright

Description: This book was first published in 1938, and is still in print, with the latest edition being the 6th (2008). It is likely that almost every serious student and researcher into number theory has consulted this book, and probably has it on their bookshelf. It was not intended to be a textbook, and is rather an introduction to a wide range of differing areas of number theory which would now almost certainly be covered in separate volumes. The writing style has long been regarded as exemplary, and the approach gives insight into a variety of areas without requiring much more than a good grounding in algebra, calculus and complex numbers.

Popular writing

Gödel, Escher, Bach

- Douglas Hofstadter

Description: Gödel, Escher, Bach: an Eternal Golden Braid is a Pulitzer Prize-winning book, first published in 1979 by Basic Books. It is a book about how the creative achievements of logician Kurt Gödel, artist M. C. Escher and composer Johann Sebastian Bach interweave. As the author states: "I realized that to me, Gödel and Escher and Bach were only shadows cast in different directions by some central solid essence. I tried to reconstruct the central object, and came up with this book."

The World of Mathematics

- James R. Newman

Description: The World of Mathematics was specially designed to make mathematics more accessible to the inexperienced. It comprises nontechnical essays on every aspect of the vast subject, including articles by and about scores of eminent mathematicians, as well as literary figures, economists, biologists, and many other eminent thinkers. Includes the work of Archimedes, Galileo, Descartes, Newton, Gregor Mendel, Edmund Halley, Jonathan Swift, John Maynard Keynes, Henri Poincaré, Lewis Carroll, George Boole, Bertrand Russell, Alfred North Whitehead, John von Neumann, and many others. In addition, an informative commentary by distinguished scholar James R. Newman precedes each essay or group of essays, explaining their relevance and context in the history and development of mathematics. Originally published in 1956, it does not include many of the exciting discoveries of the later years of the 20th century but it has no equal as a general historical survey of important topics and applications.

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Higher Dimensional Algebras (HDA)

Higher-dimensional algebra

This article is about ***higher-dimensional algebra and supercategories*** *in generalized category theory, super-category theory, and also its extensions in nonabelian algebraic topology and metamathematics*.[1]

Supercategories were first introduced in 1970,[2] and were subsequently developed for applications in theoretical physics (especially quantum field theory and topological quantum field theory) and mathematical biology or mathematical biophysics.[3]

Double groupoids, fundamental groupoids, 2-categories, categorical QFTs and TQFTs

In **higher-dimensional algebra** (**HDA**), a double groupoid is a generalisation of a one-dimensional groupoid to two dimensions,[4] and the latter groupoid can be considered as a special case of a category with all invertible arrows, or morphisms.

Double groupoids are often used to capture information about geometrical objects such as higher-dimensional manifolds (or *n*-dimensional manifolds).[5] In general, an *n*-dimensional manifold is a space that locally looks like an *n*-dimensional Euclidean space, but whose global structure may be non-Euclidean. A first step towards defining higher dimensional algebras is the concept of 2-category of higher category theory, followed by the more `geometric' concept of double category [6].[7] [8] Other pathways in HDA involve: bicategories, homomorphisms of bicategories, variable categories (*aka*, indexed, or parametrized categories), topoi, effective descent, enriched and internal categories, as well as quantum categories[9] [10] [11] and quantum double groupoids.[12] In the latter case, by considering fundamental groupoids defined via a 2-functor allows one to think about the physically interesting case of quantum fundamental groupoids (QFGs) in terms of the bicategory **Span(Groupoids)**, and then constructing 2-Hilbert spaces and 2-linear maps for manifolds and cobordisms. At the next step, one obtains cobordisms with corners via natural transformations of such 2-functors. A claim was then made that, with the gauge group SU(2), ``*the extended TQFT, or ETQFT, gives a theory equivalent to the Ponzano-Regge model of quantum gravity*";[13] similarly, the Turaev-Viro model would be then obtained with representations of SU_q(2). Therefore, according to the construction proposed by Jeffrey Morton, one can describe the state space of a gauge theory – or many kinds of quantum field theories (QFTs) and local quantum physics, in terms of the transformation groupoids given by symmetries, as for example in the case of a gauge theory, by the gauge transformations acting on states that are, in this case, connections. In the case of symmetries related to quantum groups, one would obtain structures that are representation categories of quantum groupoids,[14] instead of the 2-vector spaces that are representation categories of groupoids.

Double categories, Category of categories and Supercategories

A higher level concept is thus defined as a category of categories, or **super-category**, which generalises to higher dimensions the notion of category – regarded as any structure which is an interpretation of Lawvere's axioms of the *elementary theory of abstract categories* (ETAC).[15] [16] [17] [18] Thus, a supercategory and also a super-category, can be regarded as natural extensions of the concepts of meta-category,[19] multicategory, and multi-graph, k-partite graph, or colored graph (see a color figure, and also its definition in graph theory).

Double groupoids were first introduced by Ronald Brown in 1976, in ref.[20] and were further developed towards applications in nonabelian algebraic topology.[21] [22] [23] [24] A related, 'dual' concept is that of a double algebroid, and the more general concept of R-algebroid.

Nonabelian algebraic topology

Many of the higher dimensional algebraic structures are noncommutative and, therefore, their study is a very significant part of nonabelian category theory, and also of Nonabelian Algebraic Topology (NAAT)[25] [26] which generalises to higher dimensions ideas coming from the fundamental group.[27] Such algebraic structures in dimensions greater than 1 develop the nonabelian character of the fundamental group, and they are in a precise sense *'more nonabelian than the groups'* .[28] [29] These noncommutative, or more specifically, nonabelian structures reflect more accurately the geometrical complications of higher dimensions than the known homology and homotopy groups commonly encountered in classical algebraic topology. An important part of nonabelian algebraic topology is concerned with the properties and applications of homotopy groupoids and filtered spaces. Noncommutative double groupoids and double algebroids are only the first examples of such higher dimensional structures that are nonabelian. The new methods of Nonabelian Algebraic Topology (NAAT) *``can be applied to determine homotopy invariants of spaces, and homotopy classification of maps, in cases which include some classical results, and allow results not available by classical methods"*.[30] Cubical omega-groupoids, higher homotopy groupoids, crossed modules, crossed complexes and Galois groupoids are key concepts in developing applications related to homotopy of filtered spaces, higher dimensional space structures, the construction of the fundamental groupoid of a topos E in the general theory of topoi, and also in their physical applications in nonabelian quantum theories, and recent developments in quantum gravity, as well as categorical and topological dynamics.[31] Further examples of such applications include the generalisations of noncommutative geometry formalizations of the noncommutative standard models *via* fundamental double groupoids and spacetime structures even more general than topoi or the lower-dimensional noncommutative spacetimes encountered in several topological quantum field theories and noncommutative geometry theories of quantum gravity.

A fundamental result in NAAT is the generalised, higher homotopy van Kampen theorem proven by R. Brown which states that ``*the homotopy type of a topological space can be computed by a suitable colimit or homotopy colimit over homotopy types of its pieces*". A related example is that of van Kampen theorems for categories of covering morphisms in lextensive categories.[32] Other reports of generalisations of the van Kampen theorem include statements for 2-categories[33] and a topos of topoi [34]. Important results in HDA are also the extensions of the Galois theory in categories and variable categories, or indexed/`parametrized' categories.[35] [36] The Joyal-Tierney representation theorem for topoi is also a generalisation of the Galois theory.[37] Thus, indexing by bicategories in the sense of Benabou one also includes here the Joyal-Tierney theory.[38]

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Higher category theory

Higher category theory is the part of category theory at a *higher order*, which means that some equalities are replaced by explicit arrows in order to be able to explicitly study the structure behind those equalities.

Strict higher categories

N-categories are defined inductively using the enriched category theory: 0-categories are sets, and (n+1)-categories are categories enriched over the monoidal category of n-categories (with the monoidal structure given by finite products).[1] This construction is well defined, as shown in the article on n-categories. This concept introduces higher arrows, higher compositions and higher identities, which must well behave together. For example, the category of small categories is in fact a 2-category, with natural transformations as second degree arrows. However this concept is too strict for some purposes (for example, homotopy theory), where "weak" structures arise in the form of higher categories.[2]

Weak higher categories

In weak n-categories, the associativity and identity conditions are no longer strict (that is, they are not given by equalities), but rather are satisfied up to an isomorphism of the next level. An example in topology is the composition of paths, which is associative only up to homotopy. These isomorphisms must well behave between hom-sets and expressing this is the difficulty in the definition of weak n-categories. Weak 2-categories, also called bicategories, were the first to be defined explicitly. A particularity of these is that a bicategory with one object is exactly a monoidal category, so that bicategories can be said to be "monoidal categories with many objects." Weak 3-categories, also called tricategories, and higher-level generalizations are increasingly harder to define explicitly. Several definitions have been given, and telling when they are equivalent, and in what sense, has become a new object of study in category theory.

Quasicategories

Weak Kan complexes, or quasi-categories, are semisimplicial complexes satisfying a weak version of the Kan condition. Joyal showed that they are a good foundation for higher category theory. Recently the theory has been sistematized further by Jacob Lurie who simply call them infinity categories, though the latter term is also a generic term for all models of (infinity,k) categories for any k.

Simplicially enriched category

Simplicially enriched categories, or simplicial categories, are categories enriched over simplicial sets. However, when we look at them as a model for (infinity,1)-categories, then many categorical notions, say limits do not agree with the corresponding notions in the sense of enriched categories. The same for other enriched models like topologically enriched categories.

Topologically enriched categories

Topologically enriched categories (sometimes simply topological categories) are categories enriched over some convenient category of topological spaces, e.g. the category of compactly generated Hausdorff topological spaces.

Segal categories

These are models of higher categories introduced by Hirschowitz and Simpson in 1988 [3] , partly inspired by results of Graeme Segal in 1974.

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[1] Leinster, pp 18-19
[2] Baez, p 6
[3] André Hirschowitz, Carlos Simpson (1998), Descente pour les n-champs (Descent for n-stacks)

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- *n*lab (http://ncatlab.org/nlab/show/HomePage), the collective and open wiki notebook project on higher category theory and applications in physics, mathematics and philosophy
- Joyal's Catlab (http://ncatlab.org/joyalscatlab/show/HomePage), a wiki dedicated to polished expositions of categorical and higher categorical mathematics with proofs

External links

- John Baez Tale of *n*-Categories (http://math.ucr.edu/home/baez/week73.html)
- The n-Category Cafe (http://golem.ph.utexas.edu/category/) - a group blog devoted to higher category theory.

Duality (mathematics)

In mathematics, ***duality*** has numerous meanings, and although it is “a very pervasive and important concept in (modern) mathematics”[1] and “an important general theme that has manifestations in almost every area of mathematics”,[2] there is no single universally agreed definition that unifies all concepts of duality.[2]

Generally speaking, a duality translates concepts, theorems or mathematical structures into other concepts, theorems or structures, in a one-to-one fashion, often (but not always) by means of an involution operation: if the dual of *A* is *B*, then the dual of *B* is *A*. As involutions sometimes have fixed points, the dual of *A* is sometimes *A* itself. For example, Desargues' theorem in projective geometry is self-dual in this sense.

Many mathematical dualities between objects of two types correspond to pairings, bilinear functions from an object of one type and another object of the second type to some family of scalars. For instance, linear algebra duality corresponds in this way to bilinear maps from pairs of vector spaces to scalars, the duality between distributions and the associated test functions corresponds to the pairing in which one integrates a distribution against a test function, and Poincaré duality corresponds similarly to intersection number, viewed as a pairing between submanifolds of a given manifold.[3]

Order-reversing dualities

A particularly simple form of duality comes from order theory. The dual of a poset $P = (X, \leq)$ is the poset $P^d = (X, \geq)$ comprising the same ground set but the converse relation. Familiar examples of dual partial orders include

- the subset and superset relations $\subset$ and $\supset$ on any collection of sets,
- the *divides* and *multiple-of* relations on the integers, and
- the *descendant-of* and *ancestor-of* relations on the set of humans.

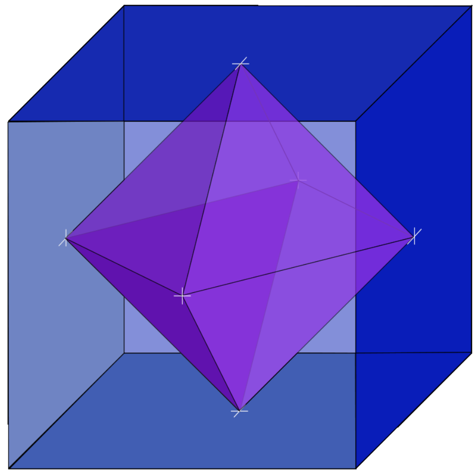
A concept defined for a partial order *P* will correspond to a *dual concept* on the dual poset P^d. For instance, a minimal element of *P* will be a maximal element of P^d: minimality and maximality are dual concepts in order theory. Other pairs of dual concepts are upper and lower bounds, lower sets and upper sets, and ideals and filters.

A particular order reversal of this type occurs in the family of all subsets of some set *S*: if $\bar{A} = S \setminus A$ denotes the complement set, then $A \subset B$ if and only if $\bar{B} \subset \bar{A}$. In topology, open sets and closed sets are dual concepts: the complement of an open set is closed, and vice versa. In matroid theory, the family of sets complementary to the independent sets of a given matroid themselves form another matroid, called the dual matroid. In logic, one may represent a truth assignment to the variables of an unquantified formula as a set, the variables that are true for the assignment. A truth assignment satisfies the formula if and only if the complementary truth assignment satisfies the De Morgan dual of its formula. The existential and universal quantifiers in logic are similarly dual.

A partial order may be interpreted as a category in which there is an arrow from *x* to *y* in the category if and only if $x \leq y$ in the partial order. The order-reversing duality of partial orders can be extended to the concept of a dual category, the category formed by reversing all the arrows in a given category. Many of the specific dualities described later are dualities of categories in this sense.

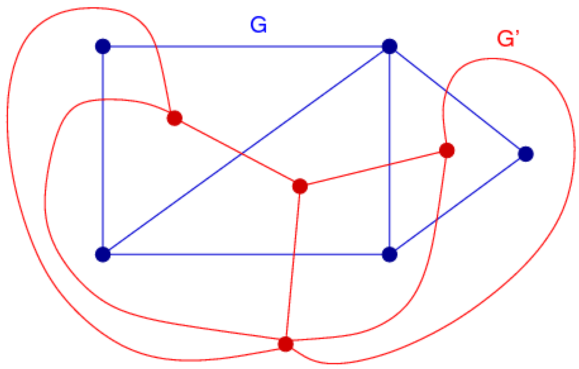
According to Artstein-Avidan and Milman,[4] [5] a *duality transform* is just an involutive antiautomorphism $\mathcal{T}$ of a partially ordered set *S*, that is, an order reversing involution $\mathcal{T} : S \to S$. Surprisingly, in several important cases these simple properties determine the transform uniquely up to some simple symmetries. If $\mathcal{T}_1, \mathcal{T}_2$ are two duality transforms then their composition is an order automorphism of *S*; thus, any two duality transforms differ only by an order automorphism. For example, all order automorphisms of a power set $S=2^R$ are induced by permutations of *R*. The papers cited above treat only sets *S* of functions on R^n satisfying some condition of convexity and prove that all order automorphisms are induced by linear or affine transformations of R^n.

Dimension-reversing dualities



The features of the cube and its dual octahedron correspond one-for-one with dimensions reversed.

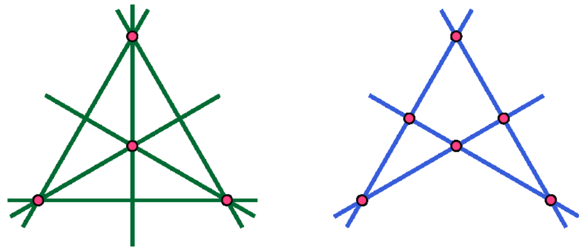
There are many distinct but interrelated dualities in which geometric or topological objects correspond to other objects of the same type, but with a reversal of the dimensions of the features of the objects. A classical example of this is the duality of the platonic solids, in which the cube and the octahedron form a dual pair, the dodecahedron and the icosahedron form a dual pair, and the tetrahedron is **self-dual**. The dual polyhedron of any of these polyhedra may be formed as the convex hull of the center points of each face of the primal polyhedron, so the vertices of the dual correspond one-for-one with the faces of the primal. Similarly, each edge of the dual corresponds to an edge of the primal, and each face of the dual corresponds to a vertex of the primal. These correspondences are incidence-preserving: if two parts of the primal polyhedron touch each other, so do the corresponding two parts of the dual polyhedron. More generally, using the concept of polar reciprocation, any convex polyhedron, or more generally any convex polytope, corresponds to a dual polyhedron or dual polytope, with an i-dimensional feature of an n-dimensional polytope corresponding to an $(n - i - 1)$-dimensional feature of the dual polytope. The incidence-preserving nature of the duality is reflected in the fact that the face lattices of the primal and dual polyhedra or polytopes are themselves order-theoretic duals. Duality of polytopes and order-theoretic duality are both involutions: the dual polytope of the dual polytope of any polytope is the original polytope, and reversing all order-relations twice returns to the original order. Choosing a different center of polarity leads to geometrically different dual polytopes, but all have the same combinatorial structure.



A planar graph, G, and its dual graph, G'.

From any three-dimensional polyhedron, one can form a planar graph, the graph of its vertices and edges. The dual polyhedron has a dual graph, a graph with one vertex for each face of the polyhedron and with one edge for every two adjacent faces. The same concept of planar graph duality may be generalized to graphs that are drawn in the plane but that do not come from a three-dimensional polyhedron, or more generally to graph embeddings on surfaces of higher genus: one may draw a dual graph by placing one vertex within each region bounded by a cycle of edges in the embedding, and drawing an edge connecting any two regions that share a boundary edge. An important example of this type comes from computational geometry: the duality for any finite set S of points in the plane between the Delaunay triangulation of S and the Voronoi diagram of S. As with dual polyhedra and dual polytopes, the duality of graphs on surfaces is a dimension-reversing involution: each vertex in the primal embedded graph corresponds to a region of the dual embedding, each edge in the primal is crossed by an edge in the dual, and each region of the primal corresponds to a vertex of the dual. The dual graph depends on how the primal graph is embedded: different planar embeddings of a single graph may lead to different dual graphs. Matroid duality is an algebraic extension of planar graph duality, in the sense that the dual matroid of the graphic matroid of a planar graph is isomorphic to the graphic matroid of the dual graph.

In topology, Poincaré duality also reverses dimensions; it corresponds to the fact that, if a topological manifold is respresented as a cell complex, then the dual of the complex (a higher dimensional generalization of the planar graph dual) represents the same manifold. In Poincaré duality, this homeomorphism is reflected in an isomorphism of the kth homology group and the $(n - k)$th cohomology group.


The complete quadrangle, a configuration of four points and six lines in the projective plane (left) and its dual configuration, the complete quadrilateral, with four lines and six points (right).

Another example of a dimension-reversing duality arises in projective geometry.[6] In the projective plane, it is possible to find geometric transformations that map each point of the projective plane to a line, and each line of the projective plane to a point, in an incidence-preserving way: in terms of the incidence matrix of the points and lines in the plane, this operation is just that of forming the transpose. Transformations of this type exist also in any higher dimension; one way to construct them is to use the same polar transformations that generate polyhedron and polytope duality. Due to this ability to replace any configuration of points and lines with a corresponding configuration of lines and points, there arises a general principle of duality in projective geometry: given any theorem in plane projective geometry, exchanging the terms "point" and "line" everywhere results in a new, equally valid theorem.[7]

The points, lines, and higher dimensional subspaces n-dimensional projective space may be interpreted as describing the linear subspaces of an $(n + 1)$-dimensional vector space; if this vector space is supplied with an inner product the transformation from any linear subspace to its perpendicular subspace is an example of a projective duality. The Hodge dual extends this duality within an inner product space by providing a canonical correspondence between the elements of the exterior algebra.

A kind of geometric duality also occurs in optimization theory, but not one that reverses dimensions. A linear program may be specified by a system of real variables (the coordinates for a point in Euclidean space $\mathbf{R}^n$), a system of linear constraints (specifying that the point lie in a halfspace; the intersection of these halfspaces is a convex polytope, the feasible region of the program), and a linear function (what to optimize). Every linear program has a dual problem with the same optimal solution, but the variables in the dual problem correspond to constraints in the primal problem and vice versa.

Duality in logic and set theory

In logic, functions or relations A and B are considered dual if $A(\neg x) = \neg B(x)$, where $\neg$ is logical negation. The basic duality of this type is the duality of the $\exists$ and $\forall$ quantifiers. These are dual because $\exists x.\neg P(x)$ and $\neg\forall x.P(x)$ are equivalent for all predicates P: if there exists an x for which P fails to hold, then it is false that P holds for all x. From this fundamental logical duality follow several others:

- A formula is said to be *satisfiable* in a certain model if there are assignments to its free variables that render it true; it is *valid* if *every* assignment to its free variables makes it true. Satisfiability and validity are dual because the invalid formulas are precisely those whose negations are satisfiable, and the unsatisfiable formulas are those whose negations are valid. This can be viewed as a special case of the previous item, with the quantifiers ranging over interpretations.
- In classical logic, the $\wedge$ and $\vee$ operators are dual in this sense, because $(\neg x \wedge \neg y)$ and $\neg(x \vee y)$ are equivalent. This means that for every theorem of classical logic there is an equivalent dual theorem. De Morgan's laws are examples. More generally, $\bigwedge(\neg x_i) = \neg \bigvee x_i$. The left side is true if and only if $\forall i.\neg x_i$, and the right side if and only if $\neg\exists i.x_i$.

- In modal logic, $\Box p$ means that the proposition p is "necessarily" true, and $\Diamond p$ that p is "possibly" true. Most interpretations of modal logic assign dual meanings to these two operators. For example in Kripke semantics, "p is possibly true" means "there exists some world W in which p is true", while "p is necessarily true" means "for all worlds W, p is true". The duality of $\Box$ and $\Diamond$ then follows from the analogous duality of ∀ and ∃. Other dual modal operators behave similarly. For example, temporal logic has operators denoting "will be true at some time in the future" and "will be true at all times in the future" which are similarly dual.

Other analogous dualities follow from these:

- Set-theoretic union and intersection are dual under the set complement operator ^{C}. That is, $(A^C \cap B^C) = (A \cup B)^C$, and more generally, $\bigcap A_\alpha^C = \left(\bigcup A_\alpha\right)^C$. This follows from the duality of ∀ and ∃: an element x is a member of $\bigcap A_\alpha^C$ if and only if $\forall\alpha.\neg x\in A_\alpha$, and is a member of $\left(\bigcup A_\alpha\right)^C$ if and only if $\neg\exists\alpha.x\in A_\alpha$.

Topology inherits a duality between open and closed subsets of some fixed topological space X: a subset U of X is closed if and only if its complement in X is open. Because of this, many theorems about closed sets are dual to theorems about open sets. For example, any union of open sets is open, so dually, any intersection of closed sets is closed. The interior of a set is the largest open set contained in it, and the closure of the set is the smallest closed set that contains it. Because of the duality, the complement of the interior of any set U is equal to the closure of the complement of U.

The collection of all open subsets of a topological space X forms a complete Heyting algebra. There is a duality, known as Stone duality, connecting sober spaces and spatial locales.

- Birkhoff's representation theorem relating distributive lattices and partial orders

Dual objects

A group of dualities can be described by endowing, for any mathematical object X, the set of morphisms Hom(X, D) into some fixed object D, with a structure similar to the one of X. This is sometimes called internal Hom. In general, this yields a true duality only for specific choices of D, in which case X^*=Hom(X, D) is referred to as the *dual* of X. It may or may not be true that the *bidual*, that is to say, the dual of the dual, $X^{**} = (X^*)^*$ is isomorphic to X, as the following example, which is underlying many other dualities, shows: the dual vector space V^* of a K-vector space V is defined as

$$V^* = \mathrm{Hom}\,(V, K).$$

The set of morphisms, i.e., linear maps, is a vector space in its own right. There is always a natural, injective map $V \to V^{**}$ given by $v \mapsto (f \mapsto f(v))$, where f is an element of the dual space. That map is an isomorphism if and only if the dimension of V is finite.

In the realm of topological vector spaces, a similar construction exists, replacing the dual by the topological dual vector space. A topological vector space that is canonically isomorphic to its bidual is called reflexive space.

The dual lattice of a lattice L is given by

$$\mathrm{Hom}(L, \mathbf{Z}),$$

which is used in the construction of toric varieties.[8] The Pontryagin dual of locally compact topological groups G is given by

$$\mathrm{Hom}(G, S^1),$$

continuous group homomorphisms with values in the circle (with multiplication of complex numbers as group operation).

Dual categories

Opposite category and adjoint functors

In another group of dualities, the objects of one theory are translated into objects of another theory and the maps between objects in the first theory are translated into morphisms in the second theory, but with direction reversed. Using the parlance of category theory, this amounts to a contravariant functor between two categories C and D:

$F: C \to D$

which for any two objects X and Y of C gives a map

$\mathrm{Hom}_C(X, Y) \to \mathrm{Hom}_D(F(Y), F(X))$

That functor may or may not be an equivalence of categories. There are various situations, where such a functor is an equivalence between the opposite category C^{op} of C, and D. Using a duality of this type, every statement in the first theory can be translated into a "dual" statement in the second theory, where the direction of all arrows has to be reversed.[9] Therefore, any duality between categories C and D is formally the same as an equivalence between C and D^{op} (C^{op} and D). However, in many circumstances the opposite categories have no inherent meaning, which makes duality an additional, separate concept.[10]

Many category-theoretic notions come in pairs in the sense that they correspond to each other while considering the opposite category. For example, Cartesian products $Y_1 \times Y_2$ and disjoint unions $Y_1 \sqcup Y_2$ of sets are dual to each other in the sense that

$\mathrm{Hom}(X, Y_1 \times Y_2) = \mathrm{Hom}(X, Y_1) \times \mathrm{Hom}(X, Y_2)$

and

$\mathrm{Hom}(Y_1 \sqcup Y_2, X) = \mathrm{Hom}(Y_1, X) \times \mathrm{Hom}(Y_2, X)$

for any set X. This is a particular case of a more general duality phenomenon, under which limits in a category C correspond to colimits in the opposite category C^{op}; further concrete examples of this are epimorphisms vs. monomorphism, in particular factor modules (or groups etc.) vs. submodules, direct products vs. direct sums (also called coproducts to emphasize the duality aspect). Therefore, in some cases, proofs of certain statements can be halved, using such a duality phenomenon. Further notions displaying related by such a categorical duality are projective and injective modules in homological algebra,[11] fibrations and cofibrations in topology and more generally model categories.[12]

Two functors $F: C \to D$ and $G: D \to C$ are adjoint if for all objects c in C and d in D

$\mathrm{Hom}_D(\mathrm{F}(c), d) \cong \mathrm{Hom}_C(c, G(d)),$

in a natural way. Actually, the correspondence of limits and colimits is an example of adjoints, since there is an adjunction

$$\mathrm{colim} : C^I \leftrightarrow C : \Delta$$

between the colimit functor that assigns to any diagram in C indexed by some category I its colimit and the diagonal functor that maps any object c of C to the constant diagramm which has c at all places. Dually,

$$\Delta : C^I \leftrightarrow C : \lim \cdot$$

Examples

For example, there is a duality between commutative rings and affine schemes: to every commutative ring A there is an affine spectrum, Spec A, conversely, given an affine scheme S, one gets back a ring by taking global sections of the structure sheaf O_S. In addition, ring homomorphisms are in one-to-one correspondence with morphisms of affine schemes, thereby there is an equivalence

(Commutative rings)^{op} ≅ (affine schemes)[13]

Compare with noncommutative geometry and Gelfand duality.

In a number of situations, the objects of two categories linked by a duality are partially ordered, i.e., there is some notion of an object "being smaller" than another one. In such a situation, a duality that respects the orderings in question is known as a Galois connection. An example is the standard duality in Galois theory (fundamental theorem of Galois theory) between field extensions and subgroups of the Galois group: a bigger field extension corresponds—under the mapping that assigns to any extension $L \supset K$ (inside some fixed bigger field Ω) the Galois group Gal(Ω / L)—to a smaller group.[14]

Pontryagin duality gives a duality on the category of locally compact abelian groups: given any such group G, the character group

$\chi(G) = \mathrm{Hom}(G, S^1)$

given by continuous group homomorphisms from G to the circle group S^1 can be endowed with the compact-open topology. Pontryagin duality states that the character group is again locally compact abelian and that

$G \cong \chi(\chi(G))$.[15]

Moreover, discrete groups correspond to compact abelian groups; finite groups correspond to finite groups. Pontryagin is the background to Fourier analysis, see below.

- Tannaka-Krein duality, a non-commutative analogue of Pontryagin duality[16]
- Gelfand duality relating commutative C*-algebras and compact Hausdorff spaces

Both Gelfand and Pontryagin duality can be deduced in a largely formal. category-theoretic way.[17]

Analytic dualities

In analysis, problems are frequently solved by passing to the dual description of functions and operators.

Fourier transform switches between functions on a vector space and its dual:

$$\hat{f}(\xi) := \int_{-\infty}^{\infty} f(x)\ e^{-2\pi i x \xi}\, dx,$$

and conversely

$$f(x) = \int_{-\infty}^{\infty} \hat{f}(\xi)\ e^{2\pi i x \xi}\, d\xi.$$

If f is an L^2-function on **R** or $\mathbf{R}^N$, say, then so is $\hat{f}$ and $f(-x) = \hat{\hat{f}}(x)$. Moreover, the transform interchanges operations of multiplication and convolution on the corresponding function spaces. A conceptual explanation of the Fourier transform is obtained by the aforementioned Pontryagin duality, applied to the locally compact groups **R** (or $\mathbf{R}^N$ etc.): any character of **R** is given by $\xi \mapsto e^{-2\pi i x \xi}$. The dualizing character of Fourier transform has many other manifestations, for example, in alternative descriptions of quantum mechanical systems in terms of coordinate and momentum representations.

- Laplace transform is similar to Fourier transform and interchanges operators of multiplication by polynomials with constant coefficient linear differential operators.
- Legendre transformation is an important analytic duality which switches between velocities in Lagrangian mechanics and momenta in Hamiltonian mechanics.

Poincaré-style dualities

Theorems showing that certain objects of interest are the dual spaces (in the sense of linear algebra) of other objects of interest are often called *dualities*. Many of these dualities are given by a bilinear pairing of two K-vector spaces

$$A \otimes B \to K.$$

For perfect pairings, there is, therefore, an isomorphism of A to the dual of B.

For example, Poincaré duality of a smooth compact complex manifold X is given by a pairing of singular cohomology with **C**-coefficients (equivalently, sheaf cohomology of the constant sheaf **C**)

$$\mathrm{H}^i(\mathrm{X}) \otimes \mathrm{H}^{2n-i}(\mathrm{X}) \to \mathbf{C},$$

where n is the (complex) dimension of X.[18] Poincaré duality can also be expressed as a relation of singular homology and de Rham cohomology, by asserting that the map

$$(\gamma, \omega) \mapsto \int_\gamma \omega$$

(integrating a differential k-form over an $2n-k$-(real)-dimensional cycle) is a perfect pairing.

The same duality pattern holds for a smooth projective variety over a separably closed field, using l-adic cohomology with $\mathbf{Q}_\ell$-coefficients instead.[19] This is further generalized to possibly singular varieties, using intersection cohomology instead, a duality called Verdier duality.[20] With increasing level of generality, it turns out, an increasing amount of technical background is helpful or necessary to understand these theorems: the modern formulation of both these dualities can be done using derived categories and certain direct and inverse image functors of sheaves, applied to locally constant sheaves (with respect to the classical analytical topology in the first case, and with respect to the étale topology in the second case).

Yet another group of similar duality statements is encountered in arithmetics: étale cohomology of finite, local and global fields (also known as Galois cohomology, since étale cohomology over a field is equivalent to group cohomology of the (absolute) Galois group of the field) admit similar pairings. The absolute Galois group $G(\mathbf{F}_q)$ of a finite field, for example, is isomorphic to $\hat{\mathbb{Z}}$, the profinite completion of **Z**, the integers. Therefore, the perfect pairing (for any G-module M)

$$\mathrm{H}^n(G, M) \times \mathrm{H}^{1-n}(G, \mathrm{Hom}\,(M, \mathbf{Q}/\mathbf{Z})) \to \mathbf{Q}/\mathbf{Z}$$ [21]

is a direct consequence of Pontryagin duality of finite groups. For local and global fields, similar statements exist (local duality and global or Poitou–Tate duality).[22]

Serre duality or coherent duality are similar to the statements above, but applies to cohomology of coherent sheaves instead.[23]

- Alexander duality

Notes

[1] Kostrikin 2001
[2] Gowers 2008, p. 187, col. 1
[3] Gowers 2008, p. 189, col. 2
[4] Artstein-Avidan & Milman 2007
[5] Artstein-Avidan & Milman 2008
[6] Veblen & Young 1965.
[7] (Veblen & Young 1965, Ch. I, Theorem 11)
[8] Fulton 1993
[9] Mac Lane 1998, Ch. II.1.
[10] (Lam 1999, §19C)
[11] Weibel (1994)
[12] Dwyer and Spaliński (1995)
[13] Hartshorne 1966, Ch. II.2, esp. Prop. II.2.3
[14] See (Lang 2002, Theorem VI.1.1) for finite Galois extensions.

[15] (Loomis 1953, p. 151, section 37D)
[16] Joyal and Street (1991)
[17] Negrepontis 1971.
[18] Griffiths & Harris 1994, p. 56
[19] Milne 1980, Ch. VI.11
[20] Iversen 1986, Ch. VII.3, VII.5
[21] Milne (2006, Example I.1.10)
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